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The Augmentation Quotients of Group Rings and the Fifth Dimension Subgroups

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1. INTRODUCTION

Let G be a group with the lower central series $G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq G_{n+1} \supseteq \cdots$. Denote by

$$W_m(G) = \sum \bigotimes_{i=1}^{m} Sp^{a_i}(G_i/G_{i+1}),$$

where $\sum$ runs over all non-negative integers $a_1, a_2, ..., a_m$ such that $\sum_{i=1}^{m} ia_i = m$, and $Sp^{a_i}(G_i/G_{i+1})$ is the a_ith symmetric power of the abelian group G_i/G_{i+1} and $Sp^0(G_i/G_{i+1}) = \mathbb{Z}$. Let $\mathbb{Z}G$ be the group ring of G over $\mathbb{Z}$, and $I = I(G)$ the augmentation ideal of $\mathbb{Z}G$. We put $Q_{m,n}(G) = I^m/I^n$ for natural integers m and n with $m < n$, and in particular $Q_m(G) = Q_{m,m+1}(G)$. Then it is well known that $Q_1(G) \cong W_1(G)$ for any group G, Losey [2] proved that $Q_2(G) \cong W_2(G)$ for any finitely generated group G, and we [9] proved that $Q_3(G) \cong W_3(G)/R_3$ for any finite group G, where R_3 is a subgroup determined exactly in $W_3(G)$, from which we determined completely the fourth dimension subgroup $D_4(G)$. Recently Sandling and Tahara [7] have proved that $Q_m(F) \cong W_m(F)$ for any free group F and any $m \geqslant 1$.

In this paper we give a canonical isomorphism

$$Q_{3,5}(G) \cong \{W_3^*(G) \oplus W_4(G)\}/R_5^*$$

for any finite group G, where $W_3^*(G)$ is the free abelian group generated by a set in one-to-one correspondence with a basis of the finite abelian group $W_3(G)$, and R_5^* is a subgroup determined exactly in $W_3^*(G) \oplus W_4(G)$. As its application we can determine the structure of $Q_4(G)$ and the fifth dimension

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subgroup $D_5(G)$. On the other hand, let G be any group. Then Sjogren [8] obtained a bound c_n for the exponent of $D_n(G)/G_n$ for all n, and in particular $c_5 = 48$, where $D_n(G)$ is the nth dimension subgroup of G. Now we can show that the exponent of $D_5(G)/G_5$ is divisible by $6 = 3!$, which will be the best bound for the exponent of $D_5(G)/G_5$.

In Section 2, following Losey [3] we list notation and preliminary results we use here. Let G be a finite group with a finite N-series $G = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n \supseteq H_{n+1} = 1$, and Λ_k be the ideal spanned over $\mathbb{Z}$ by all products $(g_1 - 1)(g_2 - 1) \cdots (g_s - 1)$ with $\sum_{j=1}^{s} w(g_j) \geqslant k$. Section 3 we find a system of canonical $\mathbb{Z}$-generators of Λ_5. In Sections 4 and 5 we determine the structure of Λ_1/Λ_3, and Λ_2/Λ_4 and Λ_3/Λ_5, respectively. As its application of the main theorem we determine completely $G \cap \{1 + \Lambda_5\}$, and we show that the exponent of $G \cap \{1 + \Lambda_5\}/H_5$ is divisible by $6 = 3!$ in the final section. Also we give a problem on dimension subgroups, which is a generalization of the work of Cohn [1], Quillen [6], and Sjogren [8] if it is affirmatively proved.

2. Notation and Preliminary Results

Let c, d be natural integers. We denote by (c, d) the greatest common divisor of c and d, and put $\binom{d}{c} = d(d-1) \cdots (d-c+1)/c(c-1) \cdots 1$.

Let A be an abelian group. Then $Sp^n(A) = A^{\otimes n}/J$, where $A^{\otimes n}$ is the nth tensor product of A and J is the subgroup of $A^{\otimes n}$ generated by all elements $x_1 \otimes x_2 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)}$ for $x_1, x_2, \ldots, x_n \in A$, and $\sigma \in S_n$, the symmetric group. $Sp^n(A)$ is called the nth *symmetric power of* A. In particular put $Sp^0(A) = \mathbb{Z}$. Denote by $x_1 \vee x_2 \vee \cdots \vee x_n$ the class in $Sp^n(A)$ to which $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ belongs.

For studies of dimension subgroups we may restrict the groups to finite p-groups by Higman's reduction theorem [4, Theorem 3.1]. Hence from now on we suppose that G is a finite group and G has a finite N-series $\mathfrak{H}$: $G = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n \supseteq H_{n+1} = 1$, that is, a series of normal subgroups H_i such that $[H_i, H_j] \subseteq H_{i+j}$ for all i, j. Then we put

$$W_m(\mathfrak{H}) = \sum \bigotimes_{i=1}^{m} Sp^{a_i}(H_i/H_{i+1}),$$

where $\sum$ runs over all non-negative integers $a_1, a_2, \ldots, a_m$ such that $\sum_{i=1}^{m} ia_i = m$. For simplicity, we put $W_m = W_m(\mathfrak{H})$ for the fixed N-series $\mathfrak{H}$. In particular if the N-series $\mathfrak{H}$ is the lower central series of G, we put $W_m(\mathfrak{H}) = W_m(G)$ as in the Introduction.

On the other hand, following Losey [3], we introduce a uniqueness basis for G relative to $\mathfrak{H}$. The N-series induces a weight function w on G: for any element x of G

$$\begin{aligned} w(x) &= k \qquad (x \in H_k - H_{k+1}) \\ &= \infty \qquad (x = 1). \end{aligned}$$

Define a family $\{A_k\}_{k=0}^{\infty}$ of ideals of $\mathbb{Z}G$ as follows. A_k is spanned over $\mathbb{Z}$ by all products $(g_1 - 1)(g_2 - 1) \cdots (g_s - 1)$ with $\sum_{j=1}^{s} w(g_j) \geqslant k$. Then A_k is an ideal of $\mathbb{Z}G$ with $A_0 = \mathbb{Z}G$, $A_1 = I$, $A_k \supseteq I^k$ and $A_i \cdot A_j \subseteq A_{i+j}$ for all $i, j \geqslant 0$. The filtration $\{A_k\}_{k=0}^{\infty}$ is called the *canonical filtration of $I(G)$ with respect to* $\mathfrak{H}$. If $x \neq 1$, define $o^*(x)$ to be the order of the coset $\bar{x} = xH_{w(x)+1}$ in $H_{w(x)}/H_{w(x)+1}$. Each quotient H_i/H_{i+1} is a finite abelian group whose operation is denoted additively, and hence there exist elements x_{i1}, x_{i2},..., $x_{i\lambda(i)}$ with $\bar{x}_{ij} = x_{ij}H_{i+1}$ such that any element $\bar{g} \in H_i/H_{i+1}$ can be uniquely written in the form

$$\bar{g} = a(1)\bar{x}_{i1} + a(2)\bar{x}_{i2} + \cdots + a(\lambda(i))\bar{x}_{i\lambda(i)},$$

where $0 \leqslant a(j) < o^*(x_{ij})$ for all j. Moreover, we choose x_{ij} so that $o^*(x_{ij})$ divides $o^*(x_{ij+1})$. Set $\Phi = \{x_{ij} | i = 1, 2,..., n, j = 1, 2,..., \lambda(i)\}$. Order Φ by setting $x_{ij} < x_{kl}$ if $i < k$ or $i = k$ and $j < l$. Then every element $g \in G$ can be uniquely written in the form

$$g = \prod_{i=1}^{n} x_{i1}^{\alpha_{i1}} x_{i2}^{\alpha_{i2}} \cdots x_{i\lambda(i)}^{\alpha_{i\lambda(i)}} \qquad (\#)$$

for some integers α_{ij} with $0 \leqslant \alpha_{ij} < o^*(x_{ij})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant \lambda(i)$, where $\prod$ runs in order of increasing i from left to right. The ordered set Φ is called a *uniqueness basis for G relative to* $\mathfrak{H}$. Put $m = \sum_{i=1}^{n} \lambda(i)$, and consider an m-sequence $\alpha = (\alpha_{ij}) = (\alpha_{11},..., \alpha_{1\lambda(1)}, \alpha_{21},..., \alpha_{2\lambda(2)},..., \alpha_{n1},..., \alpha_{n\lambda(n)})$ for non-negative integers α_{ij}. The set of all m-sequences is naturally well ordered, that is, we define $\alpha = (\alpha_{ij}) < \beta = (\beta_{ij})$ if there exist k and l with $1 \leqslant k \leqslant n$ and $1 \leqslant j \leqslant \lambda(k)$ such that $\alpha_{ij} = \beta_{ij}$ $(1 \leqslant i \leqslant k-1,\ 1 \leqslant j \leqslant \lambda(i))$, $\alpha_{kj} = \beta_{kj}$ $(1 \leqslant j \leqslant l-1)$, and $\alpha_{kl} < \beta_{kl}$. An m-sequence $\alpha = (\alpha_{ij})$ is basic if $0 \leqslant \alpha_{ij} < o^*(x_{ij})$ for all i and j. It follows from the uniqueness of the expression $(\#)$ that there is a bijection between the elements of G and the basic m-sequences. The weight W of an m-sequence $\alpha = (\alpha_{ij})$ is defined to be

$$W(\alpha) = \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant \lambda(n)}} w(x_{ij})\alpha_{ij} = \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant \lambda(n)}} i\alpha_{ij}.$$

Given an m-sequence $\alpha = (\alpha_{ij})$ we define the *proper product* $P(\alpha) \in \mathbb{Z}G$ by

$$P(\alpha) = \prod_{i=1}^{n} (x_{i1} - 1)^{\alpha_{i1}} (x_{i2} - 1)^{\alpha_{i2}} \cdots (x_{i\lambda(i)} - 1)^{\alpha_{i\lambda(i)}},$$

where $\prod$ runs in order of increasing i from left to right. If $W(\alpha) \geqslant k$, then $P(\alpha) \in \Lambda_k$. If α is a basic m-sequence, then $P(\alpha)$ is called a *basic product*. We introduce a notation for cutting an element of G modulo a desire subgroup H_i. Let $p_i(g)$ be the product of weight i in the expression (#), and $\bar{p}_i(g)$ the coset of $p_i(g)$ modulo H_{i+1}.

We have the following three lemmas.

LEMMA 2.1. *Any element $g \in G$ can be written uniquely in the form (#), and $g-1$ is represented in a $\mathbb{Z}$-linear combination of basic products as follows:*

$$g-1=\sum \prod_{i=1}^{n} \binom{\alpha_{i1}}{\beta_{i1}}\binom{\alpha_{i2}}{\beta_{i2}} \cdots \binom{\alpha_{i\lambda(i)}}{\beta_{i\lambda(i)}} (x_{i1}-1)^{\beta_{i1}} \times (x_{i2}-1)^{\beta_{i2}} \cdots (x_{i\lambda(i)}-1)^{\beta_{i\lambda(i)}},$$

where $\sum$ runs over all integers β_{ij} with $0 \leqslant \beta_{ij} \leqslant \alpha_{ij}$. In particular, for any element $x \in G$ and any natural integer d we have

$$x^d-1=\sum_{k=1}^{d}\binom{d}{k}(x-1)^k.$$

LEMMA 2.2. *If G is a group, then it follows that for any element $x, y \in G$*

$$\begin{aligned}(y-1)(x-1) &= (x-1)(y-1)+([y,x]-1)+(x-1)([y,x]-1)\\ &\quad +(y-1)([y,x]-1)+(x-1)(y-1)([y,x]-1),\end{aligned}$$

where $[y,x]=y^{-1}x^{-1}yx$.

LEMMA 2.3. *Let d be a non-negative integer. If $G=H_1 \supseteq H_2 \supseteq H_3 \supseteq H_4 \supseteq H_5=1$, then we have*

(1) $[x, hh'] = [x,h][x,h'] \qquad (x \in G,\ h, h' \in H_2)$,

(2) $$\begin{aligned}[x,y]^d &= [x,y^d][x,y,y]^{-\binom{d}{2}}[x,y,y,y]^{-\binom{d}{3}}\\ &= [x^d,y][x,y,x]^{-\binom{d}{2}}[x,y,x,x]^{-\binom{d}{3}}\end{aligned} \qquad (x, y \in G),$$

(3) $$\begin{aligned}[x,y,z]^d &= [x^d,y,z][y,x,x,z]^{\binom{d}{2}}\\ &= [x,y^d,z][y,x,y,z]^{\binom{d}{2}} \qquad (x,y,z \in G),\\ &= [x,y,z^d][y,x,z,z]^{\binom{d}{2}}\end{aligned}$$

(4) $[x,y,h][y,h,x][h,x,y]=1, \qquad (x, y \in G,\ h \in H_2)$.

Proof. (1) is clear. By induction on d and by (1) we have

$$\begin{aligned}[x, y^d] &= [x, y^{d-1}y] = [x, y][x, y^{d-1}][x, y^{d-1}, y] \\ &= [x, y]^d [x, y, y]^{\binom{d-1}{2}+d-1} [x, y, y, y]^{\binom{d-1}{3}+\binom{d-1}{2}} \\ &= [x, y]^d [x, y, y]^{\binom{d}{2}} [x, y, y, y]^{\binom{d}{3}}.\end{aligned}$$

The second part of (2) follows from the above. Part (3) follows easily from (1) and (2). Q.E.D.

3. The Structure of Λ_k $(1 \leqslant k \leqslant 5)$

It is clear that $\operatorname{rank}_{\mathbb{Z}}(\Lambda_k) = |G| - 1$ for any finite group G and all $k \geqslant 1$, and Losey [3] determined the $\mathbb{Z}$-basis of Λ_1, Λ_2 and Λ_3. We [9] gave a system of $\mathbb{Z}$-generators of Λ_4. We repeat them here, and we give a system of $\mathbb{Z}$-generators of Λ_5 to determine the structure of Λ_3/Λ_5.

For simplicity we put $\lambda(1) = s$, $\lambda(2) = t$, $\lambda(3) = u$, $\lambda(4) = v$; $o^*(x_{1i}) = d(i)$ $(1 \leqslant i \leqslant s)$, $o^*(x_{2i}) = e(i)$ $(1 \leqslant i \leqslant t)$, $o^*(x_{3i}) = f(i)$ $(1 \leqslant i \leqslant u)$, $o^*(x_{4i}) = g(i)$ $(1 \leqslant i \leqslant v)$.

Then we have

Lemma 3.1 [3, Lemma 4]. *Λ_1 has a $\mathbb{Z}$-free basis consisting of basic products distinct from* 1.

Lemma 3.2 [3, Lemma 6]. *Λ_2 has a $\mathbb{Z}$-free basis consisting of*

(1) $(x_{1i} - 1)^{d(i)}$, $1 \leqslant i \leqslant s$;

(2) $P(\alpha)$, *α basic*, $W(\alpha) \geqslant 2$.

Lemma 3.3 [3, Lemma 7]. *Λ_3 has a $\mathbb{Z}$-free basis consisting of*

(1) $(x_{1i} - 1)^{d(i)}$, $d(i) \geqslant 3$, $1 \leqslant i \leqslant s$;

(2) $(x_{2i} - 1)^{e(i)}$, $1 \leqslant i \leqslant t$;

(3) $d(i)(x_{1i} - 1)(x_{1j} - 1)$, $1 \leqslant i \leqslant j \leqslant s$;

(4) $P(\alpha)$, *α basic*, $W(\alpha) \geqslant 3$.

Lemma 3.4 [9, Lemma 4]. *Λ_4 has a system of $\mathbb{Z}$-generators consisting of*

(1) $(x_{1i} - 1)^{d(i)}$, $d(i) \geqslant 4$, $1 \leqslant i \leqslant s$;

(2) $(x_{2i} - 1)^{e(i)}$, $1 \leqslant i \leqslant t$;

(3) $(x_{3i} - 1)^{f(i)}$, $1 \leqslant i \leqslant u$;

(4) $(x_{1i} - 1)^{d(i)}(x_{1j} - 1)$, $d(i) \geqslant 3$, $1 \leqslant i \leqslant j \leqslant s$;

(5) $(x_{1i}-1)(x_{1j}-1)^{d(j)}$, $d(j)\geqslant 3$, $1\leqslant i\leqslant j\leqslant s$;

(6) $(d(i), e(j))(x_{1i}-1)(x_{2j}-1)$, $1\leqslant i\leqslant s$, $1\leqslant j\leqslant t$;

(7) $d(i)(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)$, $1\leqslant i\leqslant j\leqslant k\leqslant s$;

(8) $P(\alpha)$, *α basic*, $W(\alpha)\geqslant 4$.

Now we have the following

LEMMA 3.5. *Λ_5 has a system of $\mathbb{Z}$-generators consisting of*

(1) $(x_{1i}-1)^{d(i)}$, $d(i)\geqslant 5$, $1\leqslant i\leqslant s$;

(2) $(x_{2i}-1)^{e(i)}$, $e(i)\geqslant 3$, $1\leqslant i\leqslant t$;

(3) $(x_{3i}-1)^{f(i)}$, $1\leqslant i\leqslant u$;

(4) $(x_{4i}-1)^{g(i)}$, $1\leqslant i\leqslant v$;

(5) $(x_{1i}-1)^{d(i)}(x_{1j}-1)$, $d(i)\geqslant 4$, $1\leqslant i\leqslant j\leqslant s$;

(6) $(x_{1i}-1)(x_{1j}-1)^{d(j)}$, $d(j)\geqslant 4$, $1\leqslant i\leqslant j\leqslant s$;

(7) $(x_{1i}-1)^{d(i)}(x_{2j}-1)$, $d(i)\geqslant 3$, $1\leqslant i\leqslant s$, $1\leqslant j\leqslant t$;

(8) $(x_{1i}-1)(x_{2j}-1)^{e(j)}$, $1\leqslant i\leqslant s$, $1\leqslant j\leqslant t$;

(9) $(d(i), f(j))(x_{1i}-1)(x_{3j}-1)$, $1\leqslant i\leqslant s$, $1\leqslant j\leqslant u$;

(10) $e(i)(x_{2i}-1)(x_{2j}-1)$, $1\leqslant i\leqslant j\leqslant t$;

(11) $(x_{1i}-1)^{d(i)}(x_{1j}-1)(x_{1k}-1)$, $d(i)\geqslant 3$, $1\leqslant i\leqslant j\leqslant k\leqslant s$;

(12) $(x_{1i}-1)(x_{1j}-1)^{d(j)}(x_{1k}-1)$, $d(j)\geqslant 3$, $1\leqslant i\leqslant j\leqslant k\leqslant s$;

(13) $(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)^{d(k)}$, $d(k)\geqslant 3$, $1\leqslant i\leqslant j\leqslant k\leqslant s$;

(14) $(d(i), e(k))(x_{1i}-1)(x_{1j}-1)(x_{2k}-1)$, $1\leqslant i\leqslant j\leqslant s$, $1\leqslant k\leqslant t$;

(15) $d(i)(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)(x_{1l}-1)$, $1\leqslant i\leqslant j\leqslant k\leqslant l\leqslant s$;

(16) $P(\alpha)$, *α basic*, $W(\alpha)\geqslant 5$.

Proof. The elements of types (1)–(8), (11)–(13), and (16) clearly lie in Λ_5. We show that the elements of type (9) lie in Λ_5, the other cases being similar. The element

$$(x_{1i}-1)^{d(i)}(x_{3j}-1)=-\sum_{h=1}^{d(i)-1}\binom{d(i)}{h}(x_{1i}-1)^h(x_{3j}-1)$$
$$+(x_{1i}^{d(i)}-1)(x_{3j}-1)$$

lies in Λ_5 for $1\leqslant i\leqslant s$, $1\leqslant j\leqslant u$. Each of terms $(x_{1i}-1)^h(x_{3j}-1)$ with $h\geqslant 2$ and $(x_{1i}^{d(i)}-1)(x_{3j}-1)$ belong to Λ_5, and hence $d(i)(x_{1i}-1)(x_{3j}-1)\in\Lambda_5$. Similarly $f(j)(x_{1i}-1)(x_{3j}-1)\in\Lambda_5$. Thus, $(d(i), f(j))(x_{1i}-1)(x_{3j}-1)\in\Lambda_5$.

Next we show that the elements of types (1)–(16) span Λ_5 over $\mathbb{Z}$. Let α

be an m-sequence with $W(\alpha) \geqslant 5$, and by induction on α we may suppose that for all $\beta < \alpha$ with $W(\beta) \geqslant 5$, $P(\beta)$ is a $\mathbb{Z}$-linear combination of elements of types (1)–(16). If α is basic, then $P(\alpha)$ is of type (16), and we are done. So assume that α is not basic, then $P(\alpha) = P(\alpha_1)(x-1)^d P(\alpha_2)$, where $x \in \Phi$, $d = o^*(x)$, and $x - 1$ is not involved in $P(\alpha_2)$. If $W(\alpha_1) + W(\alpha_2) \geqslant 4$ or $w(x) \geqslant 5$, then

$$P(\alpha) = -\sum_{h=1}^{d-1} \binom{d}{h} P(\alpha_1)(x-1)^h P(\alpha_2) + P(\alpha_1)(x^d - 1) P(\alpha_2).$$

Therefore, $P(\alpha)$ is expressed as a $\mathbb{Z}$-linear combination of proper products $P(\beta)$, $\beta < \alpha$, $W(\beta) \geqslant 5$, and we are done. Thus, we may assume that $W(\alpha_1) + W(\alpha_2) \leqslant 3$ and $w(x) \leqslant 4$. Now we examine the cases corresponding to each of the possible values for $W(\alpha_1) + W(\alpha_2)$. If $W(\alpha_1) + W(\alpha_2) = 0$, then $P(\alpha) = (x-1)^d$, which is one of types (1)–(4). Suppose $W(\alpha_1) + W(\alpha_2) = 1$. Then $P(\alpha) = (x_{1i} - 1)^d (x_{1j} - 1)$ with $x = x_{1i}$, $x_{1j} \in \Phi$; or $P(\alpha) = (x_{1i} - 1)(x-1)^d$, $x_{1i} \in \Phi$. If $P(\alpha) = (x_{1i} - 1)^d (x_{1j} - 1)$, then $P(\alpha)$ is of type (5). Assume $P(\alpha) = (x_{1i} - 1)(x-1)^d$. If $w(x) = 1$, then $P(\alpha)$ is of type (6). If $w(x) = 2$, then $P(\alpha)$ is of type (8). If $w(x) = 3$, then we have $x = x_{3j}$ for some j, and $P(\alpha) = -\sum_{h=1}^{d-1} \binom{d}{h} (x_{1i} - 1)(x_{3j} - 1)^h + (x_{1i} - 1)(x_{3j}^d - 1)$ is a $\mathbb{Z}$-linear combination of elements of types (9) and (16). If $w(x) = 4$, then $P(\alpha)$ is similarly a $\mathbb{Z}$-linear combination of elements of type (16). Suppose $W(\alpha_1) + W(\alpha_2) = 2$. Then there are five cases, and we deal with the case $P(\alpha) = (x-1)^d (x_{2j} - 1)$; the other cases being similar. If $w(x) = 1$, then $x = x_{1i}$ for some i, and $P(\alpha)$ is of type (7). If $w(x) = 2$ and $x = x_{2i} < x_{2j}$, then $P(\alpha) = -\sum_{h=1}^{d-1} \binom{d}{h} (x_{2i} - 1)^h (x_{2j} - 1) + (x_{2i}^d - 1)(x_{2j} - 1)$ is a $\mathbb{Z}$-linear combination of elements of type (10) and $P(\beta)$, $\beta < \alpha$, $W(\beta) \geqslant 5$. Suppose $W(\alpha_1) + W(\alpha_2) = 3$. Then there are nine cases, and, for example, we deal with two cases $P(\alpha) = (x_{1i} - 1)^{d(i)} (x_{1j} - 1)(x_{1k} - 1)(x_{1l} - 1)$ and $P(\alpha) = (x_{1i} - 1)(x_{1j} - 1)(x_{1k} - 1)(x - 1)^d$. Let $P(\alpha) = (x_{1i} - 1)^{d(i)} (x_{1j} - 1)(x_{1k} - 1)(x_{1l} - 1)$. Then $P(\alpha) = -\sum_{h=1}^{d(i)-1} \binom{d(i)}{h} (x_{1i} - 1)^h (x_{1j} - 1)(x_{1k} - 1)(x_{1l} - 1) + (x_{1i}^{d(i)} - 1)(x_{1j} - 1)(x_{1k} - 1)(x_{1l} - 1)$ is a $\mathbb{Z}$-linear combination of elements of types (15) and (16), and $P(\beta)$, $\beta < \alpha$, $W(\beta) \geqslant 5$. Next let $P(\alpha) = (x_{1i} - 1)(x_{1j} - 1)(x_{1k} - 1)(x - 1)^d$. Then if $w(x) = 1$, then we have $x = x_{1l}$ for some l, and $P(\alpha)$ is a $\mathbb{Z}$-linear combination of elements of type (15) and $P(\beta)$, $\beta < \alpha$, $W(\beta) \geqslant 5$. If $w(x) \geqslant 2$, then $P(\alpha)$ is a $\mathbb{Z}$-linear combination of elements of type (16). Q.E.D.

4. The Structure of Λ_1/Λ_3, Λ_2/Λ_4, Λ_2/Λ_3, and Λ_3/Λ_4

We [9] determined the structure of Λ_1/Λ_2, Λ_2/Λ_3, and Λ_3/Λ_4. Now we give the structure of Λ_1/Λ_3 and Λ_2/Λ_4. Let G be a finite group with a finite

N-series $\mathfrak{H}$, and we consider the finite abelian group $W_m = W_m(\mathfrak{H})$ for each m, and denote by $W_m^* = W_m^*(\mathfrak{H})$ the free abelian group generated by a basis of W_m. Note that such basis elements will have finite order in W_m but infinite order in W_m^*; context will clarify the status of a given element.

Then we have

THEOREM 4.1. *If G is a finite group with a finite N-series $\mathfrak{H}$, then*

$$\Delta_1/\Delta_3 \cong (W_1^* \oplus W_2)/R_3^*,$$

where R_3^* is the subgroup of $W_1^* \oplus W_2$ generated by the elements $d(i)\bar{x}_{1i} + \binom{d(i)}{2}(\bar{x}_{1i} \vee \bar{x}_{1i}) - \overline{x_{1i}^{d(i)}}$ $(1 \leqslant i \leqslant s)$, and $\overline{x_{1i}^{d(i)}} = \bar{p}_2(x_{1i}^{d(i)})$.

Proof. By Lemma 3.1, any element γ of Δ_1 can be uniquely written as a $\mathbb{Z}$-linear combination of $(x_{1i}-1)$ $(1 \leqslant i \leqslant s)$, $(x_{1i}-1)(x_{1j}-1)$: basic $(1 \leqslant i \leqslant j \leqslant s)$, $x_{2i}-1$ $(1 \leqslant i \leqslant t)$, and $P(\alpha)$: basic with $W(\alpha) \geqslant 3$. Then we can define a homomorphism $\psi_1 : \Delta_1 \to (W_1^* \oplus W_2)/R_3^*$ by

$$\begin{aligned}
&\psi_1((x_{1i}-1)) = \bar{x}_{1i} + R_3^*, && 1 \leqslant i \leqslant s,\\
&\psi_1((x_{1i}-1)(x_{1j}-1)) = (\bar{x}_{1i} \vee \bar{x}_{1j}) + R_3^*, && (x_{1i}-1)(x_{1j}-1) \text{ basic},\ 1 \leqslant i \leqslant j \leqslant s,\\
&\psi_1((x_{2i}-1)) = \bar{x}_{2i} + R_3^*, && 1 \leqslant i \leqslant t,\\
&\psi_1(P(\alpha)) = R_3^*, \quad P(\alpha) \text{ basic}, && W(\alpha) \geqslant 3.
\end{aligned}$$

Then we can easily show that for all $1 \leqslant i \leqslant j \leqslant s$

$$\psi_1((x_{1i}-1)(x_{1j}-1)) = (\bar{x}_{1i} \vee \bar{x}_{1j}) + R_3^*.$$

Therefore, $\psi_1(d(i)(x_{1i}-1)(x_{1j}-1)) = d(i)(\bar{x}_{1i} \vee \bar{x}_{1j}) + R_3^* = R_3^*$. And we have easily $\psi_1((x_{1i}-1)^{d(i)}) = R_3^*$, $d(i) \geqslant 3$, $1 \leqslant i \leqslant s$, and $\psi_1((x_{2i}-1)^{e(i)}) = R_3^*$, $1 \leqslant i \leqslant t$. Thus, by Lemma 3.3 we have $\psi_1(\Delta_3) = R_3^*$. Moreover, ψ_1 is surjective. Hence ψ_1 induces a surjective homomorphism Ψ_1 from Δ_1/Δ_3 to $(W_1^* \oplus W_2)/R_3^*$. We show that Ψ_1 is injective. Let $U = \sum_{i=1}^{s} A_i(x_{1i}-1) + \sum_{1 \leqslant i \leqslant j \leqslant s}^{\text{basic}} B_{ij}(x_{1i}-1)(x_{1j}-1) + \sum_{j=1}^{t} C_j(x_{2j}-1) + \Delta_3$ be any element of $\operatorname{Ker} \Psi_1$. Put $x_{1i}^{d(i)} = \prod_{j=1}^{t} x_{2j}^{b_{ij}} x_3$ $(x_3 \in H_3)$, $1 \leqslant i \leqslant s$; then there are integers u_i $(1 \leqslant i \leqslant s)$ such that $A_i = u_i d(i)$ $(1 \leqslant i \leqslant s)$, $B_{ii} = u_i \binom{d(i)}{2}$ $(1 \leqslant i \leqslant s$, $d(i) \geqslant 3)$, $u_i = 0$ $(1 \leqslant i \leqslant s$, $d(i) = 2)$, and $C_j = -\sum_{i=1}^{s} u_i b_{ij}$ $(1 \leqslant j \leqslant t)$, and we have $B_{ij} = 0$ $(1 \leqslant i < j \leqslant s)$. Therefore,

$$U = \sum_{\substack{1 \leqslant i \leqslant s\\ d(i) \geqslant 3}} u_i \left\{ d(i)(x_{1i}-1) + \binom{d(i)}{2}(x_{1i}-1)^2 - (x_{1i}^{d(i)}-1) \right\} + \Delta_3 = \Delta_3,$$

and hence Ψ_1 is injective. Thus, Ψ_1 is an isomorphism from Δ_1/Δ_3 to $(W_1^* \oplus W_2)/R_3^*$. Q.E.D.

COROLLARY 4.2 [9, Theorem 6].

$$\Lambda_2/\Lambda_3 \cong W_2.$$

Proof. We consider the following canonical exact sequence

$$0 \longrightarrow \Lambda_2/\Lambda_3 \xrightarrow{\theta} \Lambda_1/\Lambda_3 \xrightarrow{\eta} \Lambda_1/\Lambda_2 \longrightarrow 0.$$

Then $\Lambda_2/\Lambda_3 \cong \operatorname{Ker}\eta$. If we identify $\Lambda_1/\Lambda_3 = (W_1^* \oplus W_2)/R_3^*$ and $\Lambda_1/\Lambda_2 = W_1$, then we have $\operatorname{Ker}\eta = (W_2 + R_3^*)/R_3^* \cong W_2/(W_2 \cap R_3^*) \cong W_2$. Q.E.D.

In the same way as the above we have

THEOREM 4.3. *If G is a finite group with a finite N-series $\mathfrak{H}$, then*

$$\Lambda_2/\Lambda_4 \cong (W_2^* \oplus W_3)/R_4^*,$$

where R_4^* is the subgroup of $W_2^* \oplus W_3$ generated by the following elements

(I) $e(i)\bar{x}_{2i} - \overline{x_{2i}^{e(i)}}$, $1 \leqslant i \leqslant t$;

(II) $d(i)(\bar{x}_{1i} \vee \bar{x}_{1j}) + \binom{d(i)}{2}(\bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1j}) - (\bar{x}_{1j} \otimes \overline{x_{1i}^{d(i)}}) - \overline{[x_{1i}^{d(i)}, x_{1j}]}$, $1 \leqslant i \leqslant j \leqslant s$;

(III) $d(j)(\bar{x}_{1i} \vee \bar{x}_{1j}) + \binom{d(j)}{2}(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1j}) - (\bar{x}_{1i} \oplus \overline{x_{1j}^{d(j)}})$, $1 \leqslant i \leqslant j \leqslant s$;

where $\overline{x_{1i}^{d(i)}} = \bar{p}_2(x_{1i}^{d(i)})$, $\overline{x_{21}^{e(i)}} = \bar{p}_3(x_{2i}^{e(i)})$, and $\overline{[x_{1i}^{d(i)}, x_{1j}]} = \bar{p}_3([x_{1i}^{d(i)}, x_{1j}])$.

Notation. For simplicity we denote by γ_{ij} $(1 \leqslant i < j \leqslant s)$ the elements in W_3:

$$\left\{\frac{d(j)}{d(i)}\binom{d(i)}{2}(\bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1j}) - \binom{d(j)}{2}(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1j})\right\} + \left\{(\bar{x}_{1i} \otimes \overline{x_{1j}^{d(j)}}) - \frac{d(j)}{d(i)}(\bar{x}_{1j} \otimes \overline{x_{1i}^{d(i)}})\right\} - \frac{d(j)}{d(i)}\overline{[x_{1i}^{d(i)}, x_{1j}]}.$$

Then we have the following:

COROLLARY 4.4 [9, Theorem 7].

$$\Lambda_3/\Lambda_4 \cong W_3/R_4,$$

where R_4 is the subgroup of W_3 generated by the elements γ_{ij}, $1 \leqslant i < j \leqslant s$.

5. THE STRUCTURE OF Λ_3/Λ_5 AND Λ_4/Λ_5

We [9] determined the structure of Λ_3/Λ_4. Now we want to determine the structure of Λ_3/Λ_5 in order to determine that of $G \cap (1+\Lambda_5)$. We put

$$\begin{aligned}
x_{1i}^{d(i)} &= x_{21}^{b_{i1}} x_{22}^{b_{i2}} \cdots x_{2t}^{b_{it}} x_{31}^{c_{i1}} x_{32}^{c_{i2}} \cdots x_{3u}^{c_{iu}}\, y_{4i}, \qquad y_{4i} \in H_4,\ 1 \leqslant i \leqslant s;\\
x_{2i}^{e(i)} &= x_{31}^{d_{i1}} x_{32}^{d_{i2}} \cdots x_{3u}^{d_{iu}}\, y'_{4i}, \qquad y'_{4i} \in H_4,\ 1 \leqslant i \leqslant t;\\
x_{3i}^{f(i)} &= x_{41}^{f_{i1}} x_{42}^{f_{i2}} \cdots x_{4v}^{f_{iv}}\, y_{5i}, \qquad y_{5i} \in H_5,\ 1 \leqslant i \leqslant u;\\
[x_{1i}^{d(i)}, x_{1j}] &= x_{31}^{\alpha_1^{(ij)}} x_{32}^{\alpha_2^{(ij)}} \cdots x_{3u}^{\alpha_u^{(ij)}} x_{41}^{\beta_1^{(ij)}} x_{42}^{\beta_2^{(ij)}} \cdots x_{4v}^{\beta_v^{(ij)}}\, y_5^{(ij)}, \qquad y_5^{(ij)} \in H_5,\ 1 \leqslant i < j \leqslant s.
\end{aligned}$$

We show the following with γ_{ij} in the last part Section 4, but here γ_{ij} are considered elements in W_3^*.

THEOREM 5.1. *If G is a finite group with a finite N-series $\mathfrak{H}$, then*

$$\Lambda_3/\Lambda_5 \cong (W_3^* \oplus W_4)/R_5^*,$$

where R_5^* is the subgroup of $W_3^* \oplus W_4$ generated by the following elements:

(I) $f(i)\bar{x}_{3i} - \overline{x_{3i}^{f(i)}},\ 1 \leqslant i \leqslant u;$

(II) $$\begin{aligned}
&\gamma_{ij} + \frac{d(j)}{d(i)}\binom{d(i)}{3}(\bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1j})\\
&- \binom{d(j)}{3}(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1j} \vee \bar{x}_{1j}) - \frac{d(j)}{d(i)}(\bar{x}_{1j} \otimes \overline{[x_{1i}^{d(i)}, x_{1j}]})\\
&+ (\bar{x}_{1i} \otimes \bar{p}_3(x_{1j}^{d(j)})) - \frac{d(j)}{d(i)}(\bar{x}_{1j} \otimes \bar{p}_3(x_{1i}^{d(i)}))\\
&- \frac{d(j)}{d(i)}\bar{p}_4([x_{1i}^{d(i)}, x_{1j}]),\ 1 \leqslant i < j \leqslant s;
\end{aligned}$$

(III) $$\begin{aligned}
&d(i)(\bar{x}_{1i} \otimes \bar{x}_{2k}) + \binom{d(i)}{2}\{(\bar{x}_{1i} \vee \bar{x}_{1i}) \otimes \bar{x}_{2k}\} - \{\overline{x_{1i}^{d(i)}} \vee \bar{x}_{2k}\}\\
&- \sum_{k<l} b_{il}\overline{[x_{2l}, x_{2k}]},\ 1 \leqslant i \leqslant s,\ 1 \leqslant k \leqslant t;
\end{aligned}$$

(IV) $e(k)(\bar{x}_{1i} \otimes \bar{x}_{2k}) - (\bar{x}_{1i} \otimes \overline{x_{2k}^{e(k)}}),\ 1 \leqslant i \leqslant s,\ 1 \leqslant k \leqslant t;$

(V) $$\begin{aligned}
&d(i)(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k}) + \binom{d(i)}{2}(\bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k})\\
&- \{(\bar{x}_{1j} \vee \bar{x}_{1k}) \otimes \overline{x_{1i}^{d(i)}}\} - (\bar{x}_{1j} \otimes \overline{[x_{1i}^{d(i)}, x_{1k}]})\\
&- (\bar{x}_{1k} \otimes \overline{[x_{1i}^{d(i)}, x_{1j}]}) - \overline{[x_{1i}^{d(i)}, x_{1j}, x_{1k}]},\ 1 \leqslant i \leqslant j \leqslant k \leqslant s;
\end{aligned}$$

$$\text{(VI)}\quad d(j)(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k}) + \binom{d(j)}{2} (\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1j} \vee \bar{x}_{1k})$$
$$- \{(\bar{x}_{1i} \vee \bar{x}_{1k}) \otimes \overline{x_{1j}^{d(j)}}\} - (\bar{x}_{1i} \otimes \overline{[x_{1j}^{d(j)}, x_{1k}]}),\ 1 \leqslant i \leqslant j \leqslant k \leqslant s;$$

$$\text{(VII)}\quad d(k)(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k}) + \binom{d(k)}{2} (\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k} \vee \bar{x}_{1k})$$
$$- \{(\bar{x}_{1i} \vee \bar{x}_{1j}) \otimes \overline{x_{1k}^{d(k)}}\},\ 1 \leqslant i \leqslant j < k \leqslant s;$$

where $\overline{x_{1i}^{d(i)}} = \bar{p}_2(x_{1i}^{d(i)})$, $\overline{x_{2i}^{e(i)}} = \bar{p}_3(x_{2i}^{e(i)})$, $\overline{x_{3i}^{f(i)}} = \bar{p}_4(x_{3i}^{f(i)})$, $\overline{[x_{1i}^{d(i)}, x_{1j}]} = \bar{p}_3([x_{1i}^{d(i)}, x_{1j}])$, $\overline{[x_{2l}, x_{2k}]} = \bar{p}_4([x_{2l}, x_{2k}])$, *and* $\overline{[x_{1i}^{d(i)}, x_{1j}, x_{1k}]} = \bar{p}_4([x_{1i}^{d(i)}, x_{1j}, x_{1k}])$.

Proof. We may assume that $G = H_1 \supseteq H_2 \supseteq H_3 \supseteq H_4 \supseteq H_5 = 1$ (cf. [7, Proof of Corollary 7]). We define a homomorphism ψ_3 from Λ_3 to $W_3^* \oplus W_4$ by defining it on the basis for Λ_3 given in Lemma 3.3 as follows (all quantifiers are omitted here for simplicity):

$$\begin{aligned}
(1)\quad \psi_3((x_{1i}-1)^{d(i)}) &= \bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1i}, \qquad d(i) = 3\\
&= \bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1i}, \qquad d(i) = 4,\ 1 \leqslant i \leqslant s\\
&= 0, \qquad d(i) \geqslant 5;
\end{aligned}$$

$$\begin{aligned}
(2)\quad \psi_3((x_{2i}-1)^{e(i)}) &= \bar{x}_{2i} \vee \bar{x}_{2i}, \qquad e(i) = 2\\
&= 0, \qquad e(i) \geqslant 3, \qquad 1 \leqslant i \leqslant t;
\end{aligned}$$

$$\begin{aligned}
(3)\quad & \psi_3(d(i)(x_{1i}-1)(x_{1j}-1))\\
&= -\binom{d(i)}{2}(\bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1j}) + (\bar{x}_{1j} \otimes \overline{x_{1i}^{d(i)}}) + \overline{[x_{1i}^{d(i)}, x_{1j}]}\\
&\quad - \binom{d(i)}{3}(\bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1j}) + (\bar{x}_{1j} \otimes \overline{[x_{1i}^{d(i)}, x_{1j}]})\\
&\quad + (\bar{x}_{1j} \otimes \bar{p}_3(x_{1i}^{d(i)})) + \bar{p}_4([x_{1i}^{d(i)}, x_{1j}]),\ 1 \leqslant i \leqslant j \leqslant s;
\end{aligned}$$

(4) $\psi_3((x_{1i}-1)(x_{1j}-1)(x_{1k}-1)) = \bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k}$,
basic, $1 \leqslant i \leqslant j \leqslant k \leqslant s$;

(5) $\psi_3((x_{1i}-1)(x_{2j}-1)) = \bar{x}_{1i} \otimes \bar{x}_{2j}$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant t$;

(6) $\psi_3((x_{3i}-1)) = \bar{x}_{3i}$, $1 \leqslant i \leqslant u$;

(7) $\psi_3((x_{1i}-1)(x_{1j}-1)(x_{1k}-1)(x_{1l}-1)) = \bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k} \vee \bar{x}_{1l}$,
basic, $1 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant s$;

(8) $\psi_3((x_{1i}-1)(x_{1j}-1)(x_{2k}-1)) = (\bar{x}_{1i} \vee \bar{x}_{1j}) \otimes \bar{x}_{2k}$, basic,
$1 \leqslant i \leqslant j \leqslant s$, $1 \leqslant k \leqslant t$;

(9) $\psi_3((x_{2i}-1)(x_{2j}-1)) = \bar{x}_{2i} \vee \bar{x}_{2j}$, basic, $1 \leqslant i \leqslant j \leqslant t$;

(10) $\psi_3((x_{1i}-1)(x_{3j}-1)) = \bar{x}_{1i} \otimes \bar{x}_{3j}$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant u$,

(11) $\psi_3((x_{4i}-1)) = \bar{x}_{4i}$, $1 \leqslant i \leqslant v$;

(12) $\psi_3(P(\alpha)) = 0$, basic, $W(\alpha) \geqslant 5$.

In order to prove that Λ_3/Λ_5 is canonically isomorphic to $(W_3^* \oplus W_4)/R_5^*$, we have enough to show that $\operatorname{Ker} \psi_3 \subseteq \Lambda_5$, $\psi_3(\Lambda_5) = R_5^*$, and the canonical homomorphism $\Psi_3 : \Lambda_3/\Lambda_5 \to (W_3^* \oplus W_4)/R_5^*$ induced by ψ_3 is surjective.

We begin with

LEMMA 5.2. $\operatorname{Ker} \psi_3 \subseteq \Lambda_5$.

Proof. Let U be any element of $\operatorname{Ker} \psi_3$. Assume that U is a $\mathbb{Z}$-linear combination of the elements $(x_{1i}-1)^{d(i)}$ $(1 \leqslant i \leqslant s,\ d(i) \geqslant 3)$, $(x_{2k}-1)^{e(k)}$ $(1 \leqslant k \leqslant t)$, $d(i)(x_{1i}-1)(x_{1j}-1)$ $(1 \leqslant i \leqslant j \leqslant s)$, $(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)$ $(1 \leqslant i \leqslant j \leqslant k \leqslant s$, basic$)$, $(x_{1i}-1)(x_{2k}-1)$ $(1 \leqslant i \leqslant s,\ 1 \leqslant k \leqslant t)$, $(x_{3l}-1)$ $(1 \leqslant l \leqslant u)$, $(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)(x_{1l}-1)$ $(1 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant s$, basic$)$, $(x_{1i}-1)(x_{1j}-1)(x_{2k}-1)$ $(1 \leqslant i \leqslant j \leqslant s,\ 1 \leqslant k \leqslant t$, basic$)$, $(x_{2i}-1)(x_{2j}-1)$ $(1 \leqslant i \leqslant j \leqslant t$, basic$)$, $(x_{1i}-1)(x_{3l}-1)$ $(1 \leqslant i \leqslant s,\ 1 \leqslant l \leqslant u)$, $(x_{4l}-1)$ $(1 \leqslant l \leqslant v)$, $P(\alpha)$ $(W(\alpha) \geqslant 5$, basic$)$ with coefficients A_i, B_k, C_{ij}, D_{ijk}, E_{ik}, F_l, G_{ijkl}, H_{ijk}, I_{ij}, J_{il}, K_l, L_α, respectively. Then we have easily that $C_{ij} = 0$ $(i \leqslant j,\ d(i) = 2)$, $A_i = C_{ii}\binom{d(i)}{2}$ $(d(i) = 3)$, $D_{iii} = C_{ii}\binom{d(i)}{2}$ $(d(i) \geqslant 4)$, $D_{iij} = C_{ij}\binom{d(i)}{2}$ $(i < j,\ d(i) \geqslant 3)$, $D_{ijj} = 0$ $(i < j,\ d(j) \geqslant 3)$, $D_{ijk} = 0$ $(i < j < k)$, $E_{ik} = -\sum_{h \leqslant i} C_{hi} b_{hk}$, $F_l = -\sum_{i \leqslant j} C_{ij} a_l^{(ij)}$, $C_{ij} \equiv 0 \pmod{d(i)}$ $(i \leqslant j,\ d(i) = 3)$, $A_i \equiv C_{ii}\binom{d(i)}{3} \pmod{d(i)}$ $(d(i) = 4)$, $G_{iiii} \equiv C_{ii}\binom{d(i)}{3} \pmod{d(i)}$ $(d(i) \geqslant 5)$, $G_{iiij} \equiv C_{ij}\binom{d(i)}{3} \pmod{d(i)}$ $(i < j,\ d(i) \geqslant 4)$, $G_{ijkl} \equiv 0 \pmod{d(i)}$ (otherwise), $H_{ijk} \equiv 0$ $(\operatorname{mod}(d(i), e(k)))$, $B_k \equiv 0 \pmod{e(k)}$ $(e(k) = 2)$, $I_{ij} \equiv 0 \pmod{e(i)}$ $(i \leqslant j)$, $J_{il} \equiv -\sum_{h \leqslant i} C_{hi} \alpha_l^{(hi)} - \sum_{h \leqslant i} C_{hi} c_{hl}$ $(\operatorname{mod}(d(i), f(l)))$, $K_l \equiv -\sum_{i \leqslant j} C_{ij} \beta_l^{(ij)} \equiv 0 \pmod{g(l)}$. Therefore, we have $U = -\sum_{i \leqslant j}^{d(i) \geqslant 3} C_{ij}(x_{1i}-1)^{d(i)}(x_{1j}-1) + U^*$ with some $U^* \in \Lambda_5$, and hence $U \in \Lambda_5$ since $C_{ij} \equiv 0 \pmod{d(i)}$ for $d(i) = 3$. Q.E.D.

In order to calculate the images of generators of Λ_5 in Lemma 3.5 by ψ_3 we need to calculate the images of basis elements of W_3^* and W_4 by ψ_3 from which we have that ψ_3 is surjective modulo R_5^*. First, we consider the basis elements of W_3^*: $\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k}$ $(1 \leqslant i \leqslant j \leqslant k \leqslant s)$, $\bar{x}_{1i} \otimes \bar{x}_{2j}$ $(1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant t)$, and $\bar{x}_{3i}$ $(1 \leqslant i \leqslant u)$. The second and third cases are given in (5) and (6), respectively. For the first case we have the following:

LEMMA 5.3. *For all* $1 \leqslant i \leqslant j \leqslant k \leqslant s$, *we have*

$$\psi_3((x_{1i}-1)(x_{1j}-1)(x_{1k}-1)) \equiv \bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k} \pmod{R_5^*}.$$

Proof. If $(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)$ is basic, then this is done by (4). If $(x_{1i}-1)^3=(x_{1i}-1)^{d(i)}$, then this is done by (1). If $d(i)=2$, then we have easily $\psi_3((x_{1i}-1)^2(x_{1j}-1)) \equiv \bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1j} \pmod{R_5^*}$ for all $i \leqslant j$ by (3). If $d(j)=2$, then $d(i)=2$ and we have by (II)

$$\begin{aligned}
&\psi_3((x_{1i}-1)(x_{1j}-1)^2)=\psi_3(-d(i)(x_{1i}-1)(x_{1j}-1)+(x_{1i}-1)(x_{1j}^{d(j)}-1))\\
&=\binom{d(i)}{2}(\bar{x}_{1i}\vee \bar{x}_{1i}\vee \bar{x}_{1j})-(\bar{x}_{1j}\otimes \overline{x_{1i}^{d(i)}})-[\overline{x_{1i}^{d(i)},x_{1j}}]\\
&-(\bar{x}_{1j}\otimes \overline{[x_{1i}^{d(i)},x_{1j}]})-(\bar{x}_{1j}\otimes \bar{p}_3(x_{1i}^{d(i)}))-\bar{p}_4([x_{1i}^{d(i)},x_{1j}])\\
&\quad+(\bar{x}_{1i}\otimes \overline{x_{1j}^{d(j)}})+(\bar{x}_{1i}\otimes \bar{p}_3(x_{1j}^{d(j)}))\\
&\equiv \bar{x}_{1i}\vee \bar{x}_{1j}\vee \bar{x}_{1j} \pmod{R_5^*}.
\end{aligned}$$

Q.E.D.

Next we obtain the basis elements of W_4: $\bar{x}_{1i}\vee\bar{x}_{1j}\vee\bar{x}_{1k}\vee\bar{x}_{1l}$ $(1\leqslant i\leqslant j\leqslant k\leqslant l\leqslant s)$, $(\bar{x}_{1i}\vee\bar{x}_{1j})\otimes\bar{x}_{2k}$ $(1\leqslant i\leqslant j\leqslant s,\ 1\leqslant k\leqslant t)$, $\bar{x}_{1i}\otimes\bar{x}_{3j}$ $(1\leqslant i\leqslant s,\ 1\leqslant j\leqslant u)$, $\bar{x}_{2i}\vee\bar{x}_{2j}$ $(1\leqslant i\leqslant j\leqslant t)$, and $\bar{x}_{4i}$ $(1\leqslant i\leqslant v)$. The third and fifth cases are given in (10) and (11), respectively, while the fourth case is a consequence of (2) and (9). The second case is dealt with in Lemma 5.4 and the first in Lemma 5.5.

LEMMA 5.4. *For all $1\leqslant i\leqslant j\leqslant s$ and $1\leqslant k\leqslant t$, we have*

$$\psi_3((x_{1i}-1)(x_{1j}-1)(x_{2k}-1))\equiv(\bar{x}_{1i}\vee\bar{x}_{1j})\otimes\bar{x}_{2k} \pmod{R_5^*}.$$

Proof. If $(x_{1i}-1)(x_{1j}-1)$ is basic, then this is done. If $i=j$ and $d(i)=2$, then by (III)

$$\begin{aligned}
&\psi_3((x_{1i}-1)^{d(i)}(x_{2k}-1))\\
&\quad=\psi_3(-d(i)(x_{1i}-1)(x_{2k}-1)+(x_{1i}^{d(i)}-1)(x_{2k}-1))\\
&\quad=-d(i)(\bar{x}_{1i}\otimes\bar{x}_{2k})+\sum_{l\leqslant k} b_{il}(\bar{x}_{2l}\vee\bar{x}_{2k})+\sum_{k<l} b_{il}(\bar{x}_{2l}\vee\bar{x}_{2k})\\
&\qquad+\sum_{k<l} b_{il}[x_{2l},x_{2k}]\\
&\quad=-d(i)(\bar{x}_{1i}\otimes\bar{x}_{2k})+(\overline{x_{1i}^{d(i)}}\vee\bar{x}_{2k})+\sum_{k<l} b_{il}[x_{2l},x_{2k}]\\
&\quad\equiv(\bar{x}_{1i}\vee\bar{x}_{1i})\otimes\bar{x}_{2k} \qquad \pmod{R_5^*}.
\end{aligned}$$

Q.E.D.

LEMMA 5.5. *For all $1\leqslant i\leqslant j\leqslant k\leqslant l\leqslant s$, we have*

$$\psi_3((x_{1i}-1)(x_{1j}-1)(x_{1k}-1)(x_{1l}-1))\equiv\bar{x}_{1i}\vee\bar{x}_{1j}\vee\bar{x}_{1k}\vee\bar{x}_{1l} \qquad \pmod{R_5^*}.$$

Proof. If $(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)(x_{1l}-1)$ is basic, then this is done by (7). The non-basic cases are divided into the following six.

(a) $(x_{1i}-1)^4$ with $d(i)\leqslant 4$. If $d(i)=4$, then this is done by (1). If $d(i)=3$, then

$$\begin{aligned}
&\psi_3((x_{1i}-1)^4)=\psi_3((x_{1i}-1)(x_{1i}-1)^{d(i)})\\
&=\psi_3\left(-d(i)(x_{1i}-1)^2-\binom{d(i)}{2}(x_{1i}-1)^{d(i)}+(x_{1i}-1)(x_{1i}^{d(i)}-1)\right)\\
&=\binom{d(i)}{2}(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i})-(\bar{x}_{1i}\otimes\overline{x_{1i}^{d(i)}})+(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i})\\
&\quad-(\bar{x}_{1i}\otimes\bar{p}_3(x_{1i}^{d(i)}))-\binom{d(i)}{2}(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i})+(\bar{x}_{1i}\otimes\overline{x_{1i}^{d(i)}})\\
&\quad+(\bar{x}_{1i}\otimes\bar{p}_3(x_{1i}^{d(i)}))\\
&=\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i}.
\end{aligned}$$

If $d(i)=2$, then by Lemmas 5.3 and 5.4, and hence by (V) with $i=j=k$

$$\begin{aligned}
\psi_3((x_{1i}-1)^4)&=\psi_3((x_{1i}-1)^2(x_{1i}-1)^{d(i)})\\
&=\psi_3(-d(i)(x_{1i}-1)^3+(x_{1i}-1)^2(x_{1i}^{d(i)}-1))\\
&=-d(i)(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i})+\{(\bar{x}_{1i}\vee\bar{x}_{1i})\otimes\overline{x_{1i}^{d(i)}}\}\\
&\equiv\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i}\qquad(\mathrm{mod}\ R_5^*).
\end{aligned}$$

(b) $(x_{1i}-1)^3(x_{1j}-1)$ with $d(i)\leqslant 3$, $i<j$. If $d(i)=3$, then this is done by (3). If $d(i)=2$, then by Lemma 5.3 and (V) with $i=j$

$$\begin{aligned}
&\psi_3((x_{1i}-1)(x_{1i}-1)^{d(i)}(x_{1j}-1))\\
&\quad=\psi_3(-d(i)(x_{1i}-1)^2(x_{1j}-1)+(x_{1i}-1)(x_{1i}^{d(i)}-1)(x_{1j}-1))\\
&\quad=-d(i)(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1j})+\{(\bar{x}_{1i}\vee\bar{x}_{1j})\otimes\overline{x_{1i}^{d(i)}}\}+(\bar{x}_{1i}\otimes\overline{[x_{1i}^{d(i)},x_{1j}]})\\
&\quad\equiv\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i}\wedge\bar{x}_{1j}\qquad(\mathrm{mod}\ R_5^*).
\end{aligned}$$

(c) $(x_{1i}-1)(x_{1j}-1)^3$ with $d(j)\leqslant 3$, $i<j$. If $d(j)=3$, then by (II)

$$\begin{aligned}
&\psi_3((x_{1i}-1)(x_{1j}-1)^3)\\
&\quad=\psi_3\left(-d(j)(x_{1i}-1)(x_{1j}-1)-\binom{d(j)}{2}(x_{1i}-1)(x_{1j}-1)^2\right.
\end{aligned}$$

$$+ (x_{1i}-1)(x_{1j}^{d(j)}-1))$$

$$= -\frac{d(j)}{d(i)} \left\{ \begin{array}{l} -\binom{d(i)}{2}(\bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1j}) + (\bar{x}_{1j} \otimes \overline{x_{1i}^{d(i)}}) + \overline{[x_{1i}^{d(i)}, x_{1j}]} \\ -\binom{d(i)}{3}(\bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1j}) + (\bar{x}_{1j} \otimes \bar{p}_3(x_{1i}^{d(i)})) \\ + (\bar{x}_{1j} \otimes \overline{[x_{1i}^{d(i)}, x_{1j}]}) + \bar{p}_4([x_{1i}^{d(i)}, x_{1j}]) \end{array} \right.$$

$$-\binom{d(j)}{2}(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1j}) + (\bar{x}_{1i} \otimes \overline{x_{1j}^{d(j)}}) + (\bar{x}_{1i} \otimes \bar{p}_3(x_{1j}^{d(j)}))$$

$$\equiv \bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1j} \vee \bar{x}_{1j} \qquad (\operatorname{mod} R_5^*).$$

If $d(j)=2$, then by (VI) with $j=k$,

$$\begin{aligned} &\psi_3((x_{1i}-1)(x_{1j}-1)^3) \\ &\quad = \psi_3(-d(j)(x_{1i}-1)(x_{1j}-1)^2 + (x_{1i}-1)(x_{1j}-1)(x_{1j}^{d(j)}-1)) \\ &\quad = -d(j)(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1j}) + \{(\bar{x}_{1i} \vee \bar{x}_{1j}) \otimes \overline{x_{1j}^{d(j)}}\} \\ &\quad \equiv \bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1j} \vee \bar{x}_{1j} \qquad (\operatorname{mod} R_5^*). \end{aligned}$$

(d) $(x_{1i}-1)^2(x_{1j}-1)(x_{1k}-1)$ with $d(i)=2$, $i<j<k$. Then by Lemma 5.3 and (V), we have

$$\begin{aligned} &\psi_3((x_{1i}-1)^{d(i)}(x_{1j}-1)(x_{1k}-1)) \\ &\quad = \psi_3(-d(i)(x_{1i}-1)(x_{1j}-1)(x_{1k}-1) + (x_{1i}^{d(i)}-1)(x_{1j}-1)(x_{1k}-1)) \\ &\quad = -d(i)(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k}) + \{(\bar{x}_{1j} \vee \bar{x}_{1k}) \otimes \overline{x_{1i}^{d(i)}}\} \\ &\qquad + (x_{1j} \otimes \overline{[x_{1i}^{d(i)}, x_{1k}]}) + (\bar{x}_{1k} \otimes \overline{[x_{1i}^{d(i)}, x_{1j}]}) + [x_{1i}^{d(i)}, x_{1j}, x_{1k}] \\ &\quad \equiv \bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1j} \vee \bar{x}_{1k} \qquad (\operatorname{mod} R_5^*). \end{aligned}$$

The final two cases (the square taken in the second and third positions respectively) follow as in (d). Q.E.D.

The final step in obtaining the isomorphism is to show the following:

LEMMA 5.6. $\psi_3(\Lambda_5) = R_5^*$.

Proof. First we show that $\psi_3(\Lambda_5) \subseteq R_5^*$. We examine the image of each generator of Λ_5 given in Lemma 3.5. This is done clearly for the elements of types (1)–(5), (9), (10), (15), and (16) in Lemma 3.5. We consider case (6).

By Lemma 2.1

$$\psi_3((x_{1i}-1)(x_{1j}-1)^{d(j)})$$

$$= -\frac{d(j)}{d(i)}\left\{\begin{array}{l} -\binom{d(i)}{2}(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1j})+(\bar{x}_{1j}\otimes\overline{x_{1i}^{s(i)}})+\bar{p}_4([x_{1i}^{d(i)},x_{1j}]) \\ -\binom{d(i)}{3}(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1j})+(\bar{x}_{1j}\otimes\bar{p}_3(x_{1i}^{d(i)})) \\ +(\bar{x}_{1j}\otimes\overline{[x_{1i}^{d(i)},x_{1j}]})+\bar{p}_4([x_{1i}^{d(i)},x_{1j}]) \end{array}\right\}$$

$$-\binom{d(j)}{2}(\bar{x}_{1i}\vee\bar{x}_{1j}\vee\bar{x}_{1j})-\binom{d(j)}{3}(\bar{x}_{1i}\vee\bar{x}_{1j}\vee\bar{x}_{1j}\vee\bar{x}_{1j})$$

$$+(\bar{x}_{1i}\otimes\overline{x_{1j}^{d(j)}})+(\bar{x}_{1i}\otimes\bar{p}_3(x_{1j}^{d(j)}))\equiv 0 \qquad (\text{mod } R_5^*).$$

For case (7), by Lemma 5.4 and (III)

$$\psi_3((x_{1i}-1)^{d(i)}(x_{2k}-1))$$

$$=-d(i)(\bar{x}_{1i}\otimes\bar{x}_{2k})-\binom{d(i)}{2}\{(\bar{x}_{1i}\vee\bar{x}_{1i})\otimes\bar{x}_{2k}\}+\sum_{l\leqslant k}b_{il}(\bar{x}_{2l}\vee\bar{x}_{2k})$$

$$+\sum_{k<l}b_{il}(\bar{x}_{2k}\vee\bar{x}_{2l})+\sum_{k<l}b_{il}[x_{2l},x_{2k}]$$

$$=-d(i)(\bar{x}_{1i}\otimes\bar{x}_{2k})-\binom{d(i)}{2}\{(\bar{x}_{1i}\vee\bar{x}_{1i})\otimes\bar{x}_{2k}\}-(\overline{x_{1i}^{d(i)}}\vee\bar{x}_{2k})$$

$$-\sum_{k<l}b_{il}[x_{2l},x_{2k}]\equiv 0 \qquad (\text{mod } R_5^*).$$

For case (8), similarly we have $\psi_3((x_{1i}-1)(x_{2k}-1)^{e(k)}) = -e(k)(\bar{x}_{1i}\otimes\bar{x}_{2k})+(\bar{x}_{1i}\otimes\overline{x_{2k}^{e(k)}})\equiv 0 \ (\text{mod } R_5^*)$. For case (11), we have

$$(x_{1i}-1)^{d(i)}(x_{1j}-1)(x_{1k}-1)$$

$$=-d(i)(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)-\binom{d(i)}{2}(x_{1i}-1)^2(x_{1j}-1)(x_{1k}-1)-\cdots$$

$$-d(i)(x_{1i}-1)^{d(i)-1}(x_{1j}-1)(x_{1k}-1)+(x_{1j}-1)(x_{1k}-1)(x_{1i}^{d(i)}-1)$$

$$+(x_{1j}-1)([x_{1i}^{d(i)},x_{1k}]-1)+(x_{1k}-1)([x_{1i}^{d(i)},x_{1j}]-1)$$

$$+([x_{1i}^{d(i)},x_{1j},x_{1k}]-1)+\sum A_\alpha P(\alpha), \qquad \alpha\text{: basic}, W(\alpha)\geqslant 5, A_\alpha\in\mathbb{Z},$$

and hence by (V)

$$\begin{aligned}&\psi_3((x_{1i}-1)^{d(i)}(x_{1j}-1)(x_{1k}-1))\\&\quad=-d(i)(\bar{x}_{1i}\vee\bar{x}_{1j}\vee\bar{x}_{1k})-\binom{d(i)}{2}(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1j}\vee\bar{x}_{1k})\\&\qquad+\{(\bar{x}_{1j}\vee\bar{x}_{1k})\otimes\overline{x_{1i}^{d(i)}}\}+(\bar{x}_{1j}\otimes\overline{[x_{1i}^{d(i)},x_{1k}]})+(\bar{x}_{1k}\otimes\overline{[x_{1i}^{d(i)},x_{1j}]})\\&\qquad+[x_{1i}^{d(i)},x_{1j},x_{1k}]\equiv 0 \pmod{R_5^*}.\end{aligned}$$

For case (12), similarly by (VI) we have

$$\begin{aligned}&\psi_3((x_{1i}-1)(x_{1j}-1)^{d(j)}(x_{1k}-1))\\&\quad=-d(j)(\bar{x}_{1i}\vee\bar{x}_{1j}\vee\bar{x}_{1k})-\binom{d(j)}{2}(\bar{x}_{1i}\vee\bar{x}_{1j}\vee\bar{x}_{1j}\vee\bar{x}_{1k})\\&\qquad+\{\bar{x}_{1i}\vee\bar{x}_{1k})\otimes\overline{x_{1j}^{d(j)}}\}+(\bar{x}_{1i}\otimes\overline{[x_{1j}^{d(j)},x_{1k}]})\\&\quad\equiv 0 \pmod{R_5^*}.\end{aligned}$$

For case (13), similarly by (VII) we have $\psi_3((x_{1i}-1)(x_{1j}-1)(x_{1k}-1)^{d(k)})\equiv 0 \pmod{R_5^*}$. For case (14), if $(x_{1i}-1)(x_{1j}-1)$ is basic, then this is done by (8). If $d(i)=2$, then by (III)

$$\begin{aligned}&\psi_3((d(i),e(k))(x_{1i}-1)^2(x_{2k}-1))\\&\quad=(d(i),e(k))\{-d(i)((\bar{x}_{1i}\vee\bar{x}_{1i})\otimes\bar{x}_{2k})+(\overline{x_{1i}^{d(i)}}\vee\bar{x}_{2k})\\&\qquad+\sum_{k<l}b_{il}[x_{2l},x_{2k}]\}\\&\quad=(d(i),e(k))\binom{d(i)}{2}\{(\bar{x}_{1i}\vee\bar{x}_{1i})\otimes\bar{x}_{2k}\}\equiv 0 \pmod{R_5^*}.\end{aligned}$$

Next we show that $R_5^*\subseteq\psi_3(\Lambda_5)$. We obtain each element of generators of types (I)–(VII) in turn. Developing $(x_{3i}-1)^{f(i)}$ by Lemma 2.1, we have $f(i)(x_{3i}-1)-(x_{3i}^{f(i)}-1)\in\Lambda_5$, and hence the element of type (I) is contained in $\psi_3(\Lambda_5)$. Evolving $(d(j)/d(i))(x_{1i}-1)^{d(i)}(x_{1j}-1)-(x_{1i}-1)(x_{1j}-1)^{d(j)}$ by Lemma 2.1, we have

$$\begin{aligned}&\frac{d(j)}{d(i)}\binom{d(i)}{2}(x_{1i}-1)^2(x_{1j}-1)-\binom{d(j)}{2}(x_{1i}-1)(x_{1j}-1)^2\\&\quad+(x_{1i}-1)(x_{1j}^{d(j)}-1)-\frac{d(j)}{d(i)}(x_{1j}-1)(x_{1i}^{d(i)}-1)-\frac{d(j)}{d(i)}[x_{1i}^{d(i)},x_{1j}]\\&\quad+\frac{d(j)}{d(i)}\binom{d(i)}{3}(x_{1i}-1)^3(x_{1j}-1)-\binom{d(j)}{3}(x_{1i}-1)(x_{1j}-1)^3\\&\quad-\frac{d(j)}{d(i)}(x_{1j}-1)([x_{1i}^{d(i)},x_{1j}]-1)\in\Lambda_5,\end{aligned}$$

and hence the element of type (II) is contained in $\psi_3(\Delta_5)$. Considering $(x_{1i}-1)^{d(i)}(x_{2k}-1)$ by Lemma 2.1, we have

$$d(i)(x_{1i}-1)(x_{2k}-1)+\binom{d(i)}{2}(x_{1i}-1)^2(x_{2k}-1)$$
$$-(x_{1i}^{d(i)}-1)(x_{2k}-1)\in\Delta_5,$$

and hence the element of type (III) is contained in $\psi_3(\Delta_5)$. Similarly, we have $e(k)(x_{1i}-1)(x_{2k}-1)-(x_{1i}-1)(x_{2k}^{e(k)}-1)\in\Delta_5$. Developing $(x_{1i}-1)^{d(i)}(x_{1j}-1)(x_{1k}-1)$ by Lemma 2.1, we have

$$\text{(i)}\quad d(i)(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)+\binom{d(i)}{2}(x_{1i}-1)^2(x_{1j}-1)(x_{1k}-1)$$
$$-(x_{1j}-1)(x_{1k}-1)(x_{1i}^{d(i)}-1)-(x_{1j}-1)([x_{1i}^{d(i)},x_{1k}]-1)$$
$$-(x_{1k}-1)([x_{1i}^{d(i)},x_{1j}]-1)-([x_{1i}^{d(i)},x_{1j},x_{1k}]-1)\in\Delta_5,$$

and hence the element of type (V) is contained in $\psi_3(\Delta_5)$. Similarly we have

$$\text{(ii)}\quad d(j)(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)+\binom{d(j)}{2}(x_{1i}-1)(x_{1j}-1)^2(x_{1k}-1)$$
$$-(x_{1i}-1)(x_{1k}-1)(x_{1j}^{d(j)}-1)-(x_{1i}-1)([x_{1j}^{d(j)},x_{1k}]-1)\in\Delta_5$$

and

$$\text{(iii)}\quad d(k)(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)+\binom{d(k)}{2}(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)^2$$
$$-(x_{1i}-1)(x_{1j}-1)(x_{1k}^{d(k)}-1)\in\Delta_5.$$

Here we note that relation (ii) with $i=j\leqslant k$ is equal to that of (i) with $i=j\leqslant k$, and relation (iii) with $i\leqslant j=k$ is equal to that of (ii) with $i\leqslant j=k$. Therefore, we have $R_5^*\subseteq\psi_3(\Delta_5)$ and hence $R_5^*=\psi_3(\Delta_5)$. Q.E.D.

Thus we obtain an injective canonical homomorphism Ψ_3: $\Delta_3/\Delta_5\to(W_3^*\oplus W_4)/R_5^*$ since $\operatorname{Ker}\psi_3\subseteq\Delta_5$, but as we stated just before Lemma 5.3, ψ_3 is surjective modulo R_5^* and hence Ψ_3 is surjective. Thus, we proved that the canonical homomorphism Ψ_3: $\Delta_3/\Delta_5\to(W_3^*\oplus W_4)/R_5^*$ is an isomorphism, which completed the proof of Theorem 5.1.

COROLLARY 5.7. $\Delta_4/\Delta_5\cong W_4/(W_4\cap R_5^*)$.

Proof. We consider the following canonical exact sequence

$$0\longrightarrow\Delta_4/\Delta_5\xrightarrow{\theta}\Delta_3/\Delta_5\xrightarrow{\eta}\Delta_3/\Delta_4\longrightarrow 0.$$

Then $\Delta_4/\Delta_5\cong\operatorname{Ker}\eta$, and if we identify $\Delta_3/\Delta_5\cong(W_3^*\oplus W_4)/R_5^*$ and $\Delta_3/\Delta_4\cong W_3/R_4$, we have by Lemmas 5.3, 5.4, and 5.5,

$\eta(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k} + R_5^*) = \bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k} + R_4$ $(1 \leqslant i \leqslant j \leqslant k \leqslant s)$, $\eta(\bar{x}_{1i} \otimes \bar{x}_{2k} + R_5^*) = \bar{x}_{1i} \otimes \bar{x}_{2k} + R_4$ $(1 \leqslant i \leqslant s,\ 1 \leqslant k \leqslant t)$, $\eta(\bar{x}_{3l} + R_5^*) = \bar{x}_{3l} + R_4$ $(1 \leqslant l \leqslant u)$, and $\operatorname{Ker} \eta \subseteq (W_4 + R_5^*)/R_5^*$. Let U be any element of $\operatorname{Ker} \eta$, then we may put

$$U = \sum_{i \leqslant j \leqslant k} A_{ijk}(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k}) + \sum_{i,k} B_{ik}(\bar{x}_{1i} \otimes \bar{x}_{2k}) + \sum_{l} C_l \bar{x}_{3l} + U^* + R_5^*$$

with some $U^* \in W_4$. Then there are integers u_{ij} $(1 \leqslant i < j \leqslant s)$ such that $A_{iij} \equiv u_{ij}(d(j)/d(i))\binom{d(i)}{2}$ $(\operatorname{mod} d(i))$ $(i<j)$, $A_{ijj} \equiv -u_{ij}\binom{d(j)}{2}$ $(\operatorname{mod} d(i))$ $(i<j)$, $B_{ik} \equiv \sum_{i<h} u_{ih} b_{hk} - \sum_{h<i} u_{hi}(d(i)/d(h))\, b_{hk}$ $(\operatorname{mod}(d(i), e(k)))$ and $C_l \equiv -\sum_{i<j} u_{ij}(d(j)/d(i))\, \alpha_l^{(ij)}$ $(\operatorname{mod} f(l))$, and we have $A_{iii} \equiv 0$ $(\operatorname{mod} d(i))$ and $A_{ijk} \equiv 0$ $(\operatorname{mod} d(i))$ $(i<j<k)$. Therefore, it follows that $U = \sum_{i<j} u_{ij}\gamma_{ij} + U^{**} + R_5^*$ with some $U^{**} \in W_4$, and hence $\operatorname{Ker} \eta \subseteq \langle \gamma_{ij}, W_4, R_5^* \rangle / R_5^* = (W_4 + R_5^*)/R_5^*$. Thus we have $\operatorname{Ker} \eta = (W_4 + R_5^*)/R_5^* \cong W_4/(W_4 \cap R_5^*)$. Q.E.D.

6. The Fifth Dimension Subgroups

We [9] determined the fourth dimension subgroup $G \cap \{1 + \Delta_4\}$ of any finite group through the structure of Δ_3/Δ_4. Now we determine $G \cap \{1 + \Delta_5\}$ by considering the structure of Δ_3/Δ_5. In order to consider the fifth dimension subgroups of groups, we may restrict them to finite nilpotent groups of class 4. Then we may assume that G is a finite group with an N-series $\mathfrak{H}$: $G = H_1 \supseteq H_2 \supseteq H_3 \supseteq H_4 \supseteq H_5 = 1$. Also here we use the same notation as in the first part of Section 5. Since H_3 is abelian, we write additively the multiplication in H_3.

We want to determine $G \cap \{1 + \Delta_5\}$. Since $G \cap \{1 + \Delta_3\} = H_3$, we have $G \cap \{1 + \Delta_5\} \subseteq H_3$. Let Φ_5 be the canonical homomorphism from H_3 to Δ_3/Δ_5 defined by $\Phi_5(x) = x - 1 + \Delta_5$ $(x \in H_3)$. Then $\operatorname{Ker} \Phi_5 = G \cap \{1 + \Delta_5\}$. If we identify Δ_3/Δ_5 with $(W_3^* \oplus W_4)/R_5^*$, then we have $\operatorname{Ker} \Phi_5 = \{g \in H_3 \mid \psi_3(g-1) \in R_5^*\}$. Assume that $\psi_3(g-1)$ is a $\mathbb{Z}$-linear combination of the elements of types (I)–(VII) with coefficients u_i, u_{ij}, v_{ik}, v'_{ik}, w_{ijk}, w'_{ijk} and w''_{ijk}, respectively. Considering the coefficient of $\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k}$, we have

$$\text{(A)}\quad \begin{aligned} & w_{iii} = 0,\ 1 \leqslant i \leqslant s; \\ & u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2} + w_{iij}\, d(i) + w''_{iij}\, d(j) = 0,\ 1 \leqslant i < j \leqslant s; \\ & -u_{ij}\binom{d(j)}{2} + w_{ijj}\, d(i) + w'_{ijj}\, d(j) = 0,\ 1 \leqslant i < j \leqslant s; \\ & w_{ijk}\, d(i) + w'_{ijk}\, d(j) + w''_{ijk}\, d(k) = 0,\ 1 \leqslant i < j < k \leqslant s. \end{aligned}$$

The coefficient of $\bar{x}_{1i} \otimes \bar{x}_{2k}$ implies the following:

$$\text{(B)} \quad \sum_{i<h} u_{ih} b_{hk} - \sum_{h<i} u_{hi} \frac{d(i)}{d(h)} b_{hk} + v_{ik} d(i) + v'_{ik} e(k) = 0, \qquad 1 \leqslant i \leqslant s,\ 1 \leqslant k \leqslant t.$$

Comparing with coefficient of $\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k} \vee \bar{x}_{1l}$, we have

$$\text{(C)} \quad \begin{aligned} & u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{3} + w_{iij} \binom{d(i)}{2} \equiv 0 \ (\operatorname{mod} d(i)),\ 1 \leqslant i < j \leqslant s; \\ & w_{ijj} \binom{d(i)}{2} + w''_{iij} \binom{d(j)}{2} \equiv 0 \ (\operatorname{mod} d(i)),\ 1 \leqslant i < j \leqslant s; \\ & -u_{ij} \binom{d(j)}{3} + w'_{ijj} \binom{d(j)}{2} \equiv 0 \ (\operatorname{mod} d(i)),\ 1 \leqslant i < j \leqslant s; \\ & w_{ijk} \binom{d(i)}{2},\ w'_{ijk} \binom{d(j)}{2},\ w''_{ijk} \binom{d(k)}{2} \equiv 0 \ (\operatorname{mod} d(i)), \\ & \qquad 1 \leqslant i < j < k \leqslant s. \end{aligned}$$

The coefficient of $(\bar{x}_{1i} \vee \bar{x}_{1j}) \otimes \bar{x}_{2k}$ implies the following:

$$\text{(D)} \quad \begin{aligned} & v_{ik} \binom{d(i)}{2} - \sum_{h \leqslant i} w_{hii} b_{hk} - \sum_{i<h} w''_{iih} b_{hk} \equiv 0 \\ & \qquad (\operatorname{mod}(d(i), e(k))),\ 1 \leqslant i \leqslant s,\ 1 \leqslant k \leqslant t; \\ & \sum_{h \leqslant i} w_{hij} b_{hk} + \sum_{i<h \leqslant j} w'_{ihj} b_{hk} + \sum_{j<h} w''_{ijk} b_{hk} \equiv 0 \\ & \qquad (\operatorname{mod}(d(i), e(k))),\ 1 \leqslant i < j \leqslant s,\ 1 \leqslant k \leqslant t. \end{aligned}$$

Considering the coefficient of $\bar{x}_{1i} \otimes \bar{x}_{3l}$, we have

$$\text{(E)} \quad \begin{aligned} & -\sum_{h<i} u_{hi} \frac{d(i)}{d(h)} \alpha_l^{(hi)} + \sum_{i<h} u_{ih} c_{hl} - \sum_{h<i} u_{hi} \frac{d(i)}{d(h)} c_{hl} - \sum_{k} v'_{ik} d_{kl} \\ & - \sum_{g \leqslant i \leqslant h} w_{gih} \alpha_l^{(gh)} - \sum_{g \leqslant h \leqslant i} w_{ghi} \alpha_l^{(gh)} - \sum_{i<g \leqslant h} w'_{igh} \alpha_l^{(gh)} \equiv 0 \\ & \qquad (\operatorname{mod}(d(i), f(l))),\ 1 \leqslant i \leqslant s,\ 1 \leqslant l \leqslant u. \end{aligned}$$

The coefficient of $\bar{x}_{2k} \vee \bar{x}_{2l}$ implies the following:

$$\text{(F)} \quad \begin{aligned} & \sum_i v_{ik} b_{ik} \equiv 0 \ (\operatorname{mod} e(k)),\ 1 \leqslant k \leqslant t; \\ & \sum_i v_{ik} b_{il} + \sum_i v_{il} b_{ik} \equiv 0 \ (\operatorname{mod} e(k)),\ 1 \leqslant k < l \leqslant t. \end{aligned}$$

Comparing with coefficients of $\bar{x}_{3l}$ and $\bar{x}_{4l}$, we have that the coefficient of $\bar{x}_{3l}$ in $\psi_3(\Phi_5(g))$ is $u_l f(l) - \sum_{i<j} u_{ij}(d(j)/d(i))\alpha_l^{(ij)}$, while that of $\bar{x}_{4l}$ is $-\sum_i u_i f_{il} - \sum_{i<j} u_{ij}(d(j)/d(i))\,\beta_l^{(ij)} - \sum_{i<j} v_{ik}(\sum_{j<k} b_{ik}\gamma_l^{(kj)}) - \sum_{i\leqslant j\leqslant k} w_{ijk}\,\delta_l^{(ijk)}$, where $[x_{2k}, x_{2j}] = \sum_{l=1}^{v} \gamma_l^{(kj)} x_{4l}$ and $[x_{1i}^{d(i)}, x_{1j}, x_{1k}] = \sum_{l=1}^{v} \delta_l^{(ijk)} x_{4l}$. Then we have

$$g = -\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\,[x_{1i}^{d(i)}, x_{1j}] - \sum_{i,k} v_{ik}\Bigl(\sum_{k<l} b_{il}[x_{2l}, x_{2k}]\Bigr) - \sum_{i\leqslant j\leqslant k} w_{ijk}[x_{1i}^{d(i)}, x_{1j}, x_{1k}].$$

Thus we have the following:

THEOREM 6.1. *$G \cap \{1 + \Lambda_5\}$ is equal to the subgroup generated by the elements*

$$\sum_{1\leqslant i<j\leqslant s} u_{ij}\frac{d(j)}{d(i)}\,[x_{1i}^{d(i)}, x_{1j}] + \sum_{\substack{1\leqslant i\leqslant s\\ 1\leqslant k\leqslant t}} v_{ik}\Bigl(\sum_{k<l} b_{il}[x_{2l}, x_{2k}]\Bigr) + \sum_{1\leqslant i\leqslant j\leqslant k\leqslant s} w_{ijk}[x_{1i}^{d(i)}, x_{1j}, x_{1k}]$$

for all integers u_{ij} $(1\leqslant i<j\leqslant s)$, v_{ik} $(1\leqslant i\leqslant s,\ 1\leqslant k\leqslant t)$, v'_{ik} $(1\leqslant i\leqslant s,\ 1\leqslant k\leqslant t)$, w_{ijk} $(1\leqslant i\leqslant j\leqslant k\leqslant s)$, w'_{ijk} $(1\leqslant i<j\leqslant k\leqslant s)$, and w''_{ijk} $(1\leqslant i\leqslant j<k\leqslant s)$ satisfying conditions (A)–(F).

From here on we want to prove Corollary 6.11. Lemmas 6.2–6.7 convert the six conditions (A)–(F) in turn into statements useful for the proof of Corollary 6.11.

LEMMA 6.2. *Relations* (A) *in the proof of Theorem* 6.1 *imply*

$$(\mathrm{A}_1)\quad \sum_{i<j} w_{iij}[x_{1i}^{d(i)}, x_{1j}, x_{1i}] = \sum_{i<j} w_{iij}[x_{1i}, x_{1j}, x_{1i}^{d(i)}]$$
$$= -\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}[x_{1i}, x_{1j}, x_{1i}] - \sum_{i<j} w''_{iij}[x_{1i}, x_{1j}^{d(j)}, x_{1i}]$$
$$- \sum_{i<j} w_{iij}\binom{d(i)}{2}[x_{1j}, x_{1i}, x_{1i}, x_{1i}]$$
$$- \sum_{i<j} w''_{iij}\binom{d(j)}{2}[x_{1j}, x_{1i}, x_{1j}, x_{1i}],$$

$$(\mathrm{A}_2)\quad \sum_{i<j} w_{ijj}[x_{1i}^{d(i)}, x_{1j}, x_{1j}]$$
$$= \sum_{i<j} u_{ij}\binom{d(j)}{2}[x_{1i}, x_{1j}, x_{1j}] - \sum_{i<j} w'_{ijj}[x_{1i}, x_{1j}^{d(j)}, x_{1j}]$$

$$- \sum_{i<j} w_{ijj} \binom{d(i)}{2} [x_{1j}, x_{1i}, x_{1i}, x_{1j}]$$
$$- \sum_{i<j} w'_{ijj} \binom{d(j)}{2} [x_{1j}, x_{1i}, x_{1j}, x_{1j}].$$

Proof. By Lemma 2.3 and (A) we have

$$\sum_{i<j} w_{iij}[x_{1i}^{d(i)}, x_{1j}, x_{1i}]$$
$$= \sum_{i<j} w_{iij} \left\{ d(i)[x_{1i}, x_{1j}, x_{1i}] - \binom{d(i)}{2} [x_{1j}, x_{1i}, x_{1i}, x_{1i}] \right\}$$
$$= - \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}] - \sum_{i<j} w''_{iij} d(j)[x_{1i}, x_{1j}, x_{1i}]$$
$$- \sum_{i<j} w_{iij} \binom{d(i)}{2} [x_{1j}, x_{1i}, x_{1i}, x_{1i}]$$
$$= - \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}] - \sum_{i<j} w''_{iij}[x_{1i}, x_{1j}^{d(j)}, x_{1i}]$$
$$- \sum_{i<j} w_{iij} \binom{d(i)}{2} [x_{1j}, x_{1i}, x_{1i}, x_{1i}]$$
$$- \sum_{i<j} w''_{iij} \binom{d(j)}{2} [x_{1j}, x_{1i}, x_{1j}, x_{1i}].$$

Similarly we have (A_2). Q.E.D.

LEMMA 6.3. *From relations* (B) *in the proof of Theorem* 6.1 *it follows that*

$$\sum_{i<j} u_{ij}[x_{1i}, p_3(x_{1j}^{d(j)})] + \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} [p_3(x_{1i}^{d(i)}), x_{1j}]$$
$$+ \sum_{i,k} v'_{ik}[x_{2k}^{e(k)}, x_{1i}] - 2 \sum_{i<j} u_{ij}[x_{1i}, x_{1j}^{d(j)}]$$
$$- \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}] + \sum_{i<j} u_{ij} \binom{d(j)}{2} [x_{1i}, x_{1j}, x_{1j}]$$
$$+ \sum_{i,k} v_{ik} d(i)[x_{2k}, x_{1i}] - \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{3} [x_{1i}, x_{1j}, x_{1i}, x_{1i}]$$
$$+ \sum_{i<j} u_{ij} \binom{d(j)}{3} [x_{1i}, x_{1j}, x_{1j}, x_{1j}] = 0$$

and

$$\sum_{i<j} u_{ij}\, d(j)[x_{1i}, x_{1j}, x_{1i}] + \sum_{i<j} u_{ij}\, d(j)[x_{1i}, x_{1j}, x_{1j}]$$
$$- \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{2} [x_{1j}, x_{1i}, x_{1i}, x_{1j}]$$
$$+ \sum_{i<j} u_{ij} \binom{d(j)}{2} [x_{1i}, x_{1j}, x_{1j}, x_{1i}] = 0.$$

Proof. Since $\sum_k b_{hk}[x_{2k}, x_{1i}] = [x_{1h}^{d(h)}, x_{1i}] - [p_3(x_{1h}^{d(h)}), x_{1i}]$, summing up (B) multiplied by $[x_{2k}, x_{1i}]$ for all k and i, we have

$$\sum_{i<j} u_{ij}[x_{1j}^{d(j)}, x_{1i}] - \sum_{i<j} u_{ij}[p_3(x_{1j}^{d(j)}), x_{1i}] - \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}]$$
$$+ \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} [p_3(x_{1i}^{d(i)}), x_{1j}] + \sum_{i,k} v_{ik}\, d(i)[x_{2k}, x_{1i}]$$
$$+ \sum_{i,k} v'_{ik}[x_{2k}^{e(k)}, x_{1i}] = 0.$$

On the other hand, by Lemma 2.3

$$\frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}] = [x_{1i}, x_{1j}^{d(j)}] + \frac{d(j)}{d(i)} \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}]$$
$$- \binom{d(j)}{2} [x_{1i}, x_{1j}, x_{1j}] + \frac{d(j)}{d(i)} \binom{d(i)}{3} [x_{1i}, x_{1j}, x_{1i}, x_{1i}]$$
$$- \binom{d(j)}{3} [x_{1i}, x_{1j}, x_{1j}, x_{1j}],$$

and hence we have the first desired equation. To obtain the second one, we sum up (B) multiplied by $[x_{2k}, x_{1i}, x_{1i}]$ for all k and i, and hence

$$\sum_{i<j} u_{ij}[x_{1j}^{d(j)}, x_{1i}, x_{1i}] - \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1j}] = 0.$$

Therefore by Lemma 2.3 we have the second desired equation. Q.E.D.

LEMMA 6.4. *Relations* (C) *in the proof of Theorem* 6.1 *imply*

$$u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{3} [x_{1i}, x_{1j}, x_{1i}, x_{1i}] + w_{iij} \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}, x_{1i}] = 0,$$
$$1 \leqslant i < j \leqslant s;$$

$$w_{ijj}\binom{d(i)}{2}[x_{1j},x_{1i},x_{1i},x_{1j}]+w''_{iij}\binom{d(j)}{2}[x_{1j},x_{1i},x_{1i},x_{1j}]=0,$$
$$1\leqslant i<j\leqslant s;$$
$$-u_{ij}\binom{d(j)}{3}[x_{1i},x_{1j},x_{1j},x_{1j}]+w'_{ijj}\binom{d(j)}{2}[x_{1i},x_{1j},x_{1j},x_{1j}]=0,$$
$$1\leqslant i<j\leqslant s;$$
$$w_{ijk}\binom{d(i)}{2}[x_{1k},x_{1j},x_{1i},x_{1i}]=w'_{ijk}\binom{d(j)}{2}[x_{1j},x_{1i},x_{1j},x_{1k}]$$
$$=w''_{ijk}\binom{d(k)}{2}[x_{1k},x_{1j},x_{1k},x_{1i}]=0,\qquad 1\leqslant i<j<k\leqslant s.$$

Proof. Clear.

LEMMA 6.5. *From Relations* (D) *in the proof of Theorem* 6.1 *it follows that*

$$(\mathrm{D}_1)\quad \sum_{i<j} w_{ijj}[x_{1i}^{d(i)},x_{1j},x_{1j}]=\sum_{i,k} v_{ik}\binom{d(i)}{2}[x_{2k},x_{1i},x_{1i}]$$
$$-\sum_{i<j} w''_{iij}[x_{1j}^{d(j)},x_{1i},x_{1i}],$$

$$(\mathrm{D}_2)\quad \sum_{i<j} w'_{ijj}[x_{1j}^{d(j)},x_{1i},x_{1j}]=\sum_{i<j} w'_{ijj}[x_{1j},x_{1i},x_{1j}^{d(j)}]$$
$$=\sum_{i<j<k} w'_{ijk}\left\{\frac{d(j)}{d(i)}[x_{1i}^{d(i)},x_{1j},x_{1k}]-[x_{1j}^{d(j)},x_{1i},x_{1k}]\right\}$$
$$+\sum_{i<j<k} w''_{ijk}\left\{\frac{d(k)}{d(i)}[x_{1i}^{d(i)},x_{1j},x_{1k}]-[x_{1k}^{d(k)},x_{1i},x_{1j}]\right\},$$

$$(\mathrm{D}_3)\quad \sum_{i<j} w_{iij}[x_{1i}^{d(i)},x_{1j},x_{1i}]$$
$$=\sum_{i<j<k} w'_{ijk}\left\{\frac{d(j)}{d(i)}[x_{1i}^{d(i)},x_{1k},x_{1j}]-[x_{1j}^{d(j)},x_{1k},x_{1i}]\right\}$$
$$+\sum_{i<j<k} w''_{ijk}\left\{\frac{d(k)}{d(i)}[x_{1i}^{d(i)},x_{1k},x_{1j}]-[x_{1k}^{d(k)},x_{1j},x_{1i}]\right\}.$$

Proof. Summing up the first relation multiplied by $[x_{2k},x_{1i},x_{1i}]$ for all k and i, we have

$$\sum_{i,k} v_{ik}\binom{d(i)}{2}[x_{2k},x_{1i},x_{1i}]-\sum_{i<j} w_{ijj}[x_{1i}^{d(i)},x_{1j},x_{1j}]$$
$$-\sum_{i<j} w''_{iij}[x_{1j}^{d(j)},x_{1i},x_{1i}]=0,$$

and hence

$$\sum_{i<j} w_{ijj}[x_{1i}^{d(i)}, x_{1j}, x_{1j}] = \sum_{i,k} v_{ik} \binom{d(i)}{2} [x_{2k}, x_{1i}, x_{1i}] - \sum_{i<j} w''_{iij}[x_{1j}^{d(j)}, x_{1i}, x_{1i}].$$

Similarly summing up the second relation multiplied by $[x_{2k}, x_{1i}, x_{1j}]$ and $[x_{2k}, x_{1j}, x_{1i}]$ for all k and i, respectively, we have

$$(1) \quad \sum_{i<j<k} w_{ijk}[x_{1i}^{d(i)}, x_{1j}, x_{1k}] + \sum_{i<j\leqslant k} w'_{ijk}[x_{1j}^{d(j)}, x_{1i}, x_{1k}] + \sum_{i<j<k} w''_{ijk}[x_{1k}^{d(k)}, x_{1i}, x_{1j}] = 0$$

$$(2) \quad \sum_{i\leqslant j<k} w_{ijk}[x_{1i}^{d(i)}, x_{1k}, x_{1j}] + \sum_{i<j<k} w'_{ijk}[x_{1j}^{d(j)}, x_{1k}, x_{1i}] + \sum_{i<j<k} w''_{ijk}[x_{1k}^{d(k)}, x_{1j}, x_{1i}] = 0.$$

Since $w_{ijk} = -w'_{ijk}(d(j)/d(i)) - w''_{ijk}(d(k)/d(i))$ $(i<j<k)$, by (1) it follows that

$$\sum_{i<j} w'_{ijj}[x_{1j}^{d(j)}, x_{1i}, x_{1j}] = \sum_{i<j} w'_{ijj}[x_{1j}, x_{1i}, x_{1j}^{d(j)}]$$
$$= \sum_{i<j<k} w'_{ijk} \left\{ \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1k}] - [x_{1j}^{d(j)}, x_{1i}, x_{1k}] \right\}$$
$$+ \sum_{i<j<k} w''_{ijk} \left\{ \frac{d(k)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1k}] - [x_{1k}^{d(k)}, x_{1i}, x_{1j}] \right\}.$$

Similarly by (2) we have (D_3). Q.E.D.

LEMMA 6.6. *Relations* (E) *in the proof of Theorem* 6.1 *imply*

$$\sum_{i<j} u_{ij} \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1j}] + \sum_{i<j} u_{ij}[x_{1i}, p_3(x_{1j}^{d(j)})]$$
$$+ \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} [p_3(x_{1i}^{d(i)}), x_{1j}] + \sum_{i,k} v'_{ik}[x_{2k}^{e(k)}, x_{1i}]$$
$$+ \sum_{i\leqslant j\leqslant k} w_{ijk}[x_{1i}^{d(i)}, x_{1k}, x_{1j}] + \sum_{i\leqslant j\leqslant k} w_{ijk}[x_{1i}^{d(i)}, x_{1j}, x_{1k}]$$
$$+ \sum_{i<j\leqslant k} w'_{ijk}[x_{1j}^{d(j)}, x_{1k}, x_{1i}] = 0.$$

Proof. Summing up (E) multiplied by $[x_{3l}, x_{1i}]$ for all l and i, we have the desired equation. Q.E.D.

LEMMA 6.7. *From Relations* (F) *in the proof of Theorem* 6.1 *it follows that*

$$\sum_{l,k} v_{lk}[x_{2k}, x_{1l}^{d(l)}] = -2\sum_{i,k} v_{ik}\left(\sum_{k<l} b_{il}[x_{2l}, x_{2k}]\right).$$

Proof. Summing up the second part of (F) multiplied by $[x_{2l}, x_{2k}]$ for all $k < l$, then we have

$$\sum_{k<l}\left(\sum_i v_{ik}b_{il} + \sum_i v_{il}b_{ik}\right)[x_{2l}, x_{2k}] = 0.$$

On the other hand,

$$\begin{aligned}\sum_{l,k} v_{lk}[x_{2k}, x_{1l}^{d(l)}] &= \sum_{k<l}\left(\sum_i v_{il}b_{ik} - \sum_i v_{ik}b_{il}\right)[x_{2l}, x_{2k}]\\ &= -2\sum_{i,k} v_{ik}\left(\sum_{k<l} b_{il}[x_{2l}, x_{2k}]\right).\end{aligned}$$ Q.E.D.

By Lemmas 6.2 and 6.5 we have the following:

LEMMA 6.8.

$$\begin{aligned}&\sum_{i<j} w''_{ij}[x_{1i}, x_{1j}^{d(j)}, x_{1i}]\\ &= -\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}[x_{1i}, x_{1j}, x_{1i}]\\ &\quad -\sum_{i<j<k} w'_{ijk}\left\{\frac{d(j)}{d(i)}[x_{1i}^{d(i)}, x_{1k}, x_{1j}] - [x_{1j}^{d(j)}, x_{1k}, x_{1i}]\right\}\\ &\quad -\sum_{i<j<k} w''_{ijk}\left\{\frac{d(k)}{d(i)}[x_{1i}^{d(i)}, x_{1k}, x_{1j}] - [x_{1k}^{d(k)}, x_{1j}, x_{1i}]\right\}\\ &\quad -\sum_{i<j} w_{ij}\binom{d(i)}{2}[x_{1j}, x_{1i}, x_{1i}, x_{1i}]\\ &\quad -\sum_{i<j} w''_{ij}\binom{d(j)}{2}[x_{1j}, x_{1i}, x_{1j}, x_{1i}]\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i<j} w_{ijj}[x_{1i}^{d(i)}, x_{1j}, x_{1j}] \\
&\quad = \sum_{i<j} u_{ij} \binom{d(j)}{2} [x_{1i}, x_{1j}, x_{1j}] \\
&\quad\quad + \sum_{i<j<k} w'_{ijk} \left\{ \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1k}] - [x_{1j}^{d(j)}, x_{1i}, x_{1k}] \right\} \\
&\quad\quad + \sum_{i<j<k} w''_{ijk} \left\{ \frac{d(k)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1k}] - [x_{1k}^{d(k)}, x_{1i}, x_{1j}] \right\} \\
&\quad\quad - \sum_{i<j} w_{ijj} \binom{d(i)}{2} [x_{1j}, x_{1i}, x_{1i}, x_{1j}] \\
&\quad\quad - \sum_{i<j} w'_{ijj} \binom{d(j)}{2} [x_{1j}, x_{1i}, x_{1j}, x_{1j}].
\end{aligned}$$

LEMMA 6.9.

$$\begin{aligned}
&3 \sum_{i<j<k} w'_{ijk}[x_{1i}, x_{1j}^{d(j)}, x_{1k}] - 3 \sum_{i<j<k} w''_{ijk}[x_{1k}^{d(k)}, x_{1i}, x_{1j}] \\
&\quad + \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}] + \sum_{i<j} u_{ij} \binom{d(j)}{2} [x_{1i}, x_{1j}, x_{1j}] \\
&\quad - \sum_{i,k} v_{ik} \binom{d(i)}{2} [x_{2k}, x_{1i}, x_{1i}] + \sum_{i<j} w_{iij} \binom{d(i)}{2} [x_{1j}, x_{1i}, x_{1i}, x_{1i}] \\
&\quad - \sum_{i<j} w'_{ijj} \binom{d(j)}{2} [x_{1j}, x_{1i}, x_{1j}, x_{1j}] = 0.
\end{aligned}$$

Proof. Substituting $\sum_{i<j} w''_{iij}[x_{1i}, x_{1j}^{d(j)}, x_{1i}]$ and $\sum_{i<j} w_{ijj}[x_{1i}^{d(i)}, x_{1j}, x_{1j}]$ of Lemma 6.8 in (D_1) of Lemma 6.5, we have

$$\begin{aligned}
0 = &\sum_{i<j} u_{ij} \binom{d(j)}{2} [x_{1i}, x_{1j}, x_{1j}] + \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}] \\
&- \sum_{i,k} v_{ik} \binom{d(i)}{2} [x_{2k}, x_{1i}, x_{1i}] + \sum_{i<j} w_{iij} \binom{d(i)}{2} [x_{1j}, x_{1i}, x_{1i}, x_{1i}] \\
&+ \sum_{i<j} w''_{iij} \binom{d(j)}{2} [x_{1j}, x_{1i}, x_{1j}, x_{1i}] - \sum_{i<j} w_{ijj} \binom{d(i)}{2} [x_{1j}, x_{1i}, x_{1i}, x_{1j}] \\
&- \sum_{i<j} w'_{ijj} \binom{d(j)}{2} [x_{1j}, x_{1i}, x_{1j}, x_{1j}]
\end{aligned}$$

$$+\sum_{i<j<k} w'_{ijk}\left\{\begin{array}{l}\dfrac{d(j)}{d(i)}[x_{1i}^{d(i)}, x_{1k}, x_{1j}] - [x_{1j}^{d(j)}, x_{1k}, x_{1i}]\\ \dfrac{d(j)}{d(i)}[x_{1i}^{d(i)}, x_{1j}, x_{1k}] - [x_{1j}^{d(j)}, x_{1i}, x_{1k}]\end{array}\right\}$$

$$+\sum_{i<j<k} w''_{ijk}\left\{\begin{array}{l}\dfrac{d(k)}{d(i)}[x_{1i}^{d(i)}, x_{1k}, x_{1j}] - [x_{1k}^{d(k)}, x_{1j}, x_{1i}]\\ \dfrac{d(k)}{d(i)}[x_{1i}^{d(i)}, x_{1j}, x_{1k}] - [x_{1k}^{d(k)}, x_{1i}, x_{1j}]\end{array}\right\}.$$

On the other hand, by Lemma 2.3(3) and Theorem 6.1(C) we have

$$w'_{ijk}\frac{d(j)}{d(i)}[x_{1i}^{d(i)}, x_{1k}, x_{1j}] = w'_{ijk}[x_{1i}, x_{1k}, x_{1j}^{d(j)}]$$

and similar equations for $w'_{ijk}(d(j)/d(i))[x_{1i}^{d(i)}, x_{1j}, x_{1k}]$, $w''_{ijk}(d(k)/d(i))\,[x_{1i}^{d(i)}, x_{1k}, x_{1j}]$, and $w''_{ijk}(d(k)/d(i))[x_{1i}^{d(i)}, x_{1j}, x_{1k}]$. Then by Lemma 2.3, the last two terms of the above equation is equal to

$$3\sum_{i<j<k} w'_{ijk}[x_{1i}, x_{1j}^{d(j)}, x_{1k}] - 3\sum_{i<j<k} w''_{ijk}[x_{1k}^{d(k)}, x_{1i}, x_{1j}].$$

Therefore

$$\begin{aligned}0 = {} & 3\sum_{i<j<k} w'_{ijk}[x_{1i}, x_{1j}^{d(j)}, x_{1k}] - 3\sum_{i<j<k} w''_{ijk}[x_{1k}^{d(k)}, x_{1i}, x_{1j}]\\ & + \sum_{i<j} u_{ij}\frac{d(j)}{d(i)}[x_{1i}, x_{1j}, x_{1i}] + \sum_{i<j} u_{ij}\binom{d(j)}{2}[x_{1i}, x_{1j}, x_{1j}]\\ & - \sum_{i,k} v_{ik}\binom{d(i)}{2}[x_{2k}, x_{1i}, x_{1i}] + \sum_{i<j} w_{iij}\binom{d(i)}{2}[x_{1j}, x_{1i}, x_{1i}, x_{1i}]\\ & - \sum_{i<j} w'_{ijj}\binom{d(j)}{2}[x_{1j}, x_{1i}, x_{1j}, x_{1j}] - \sum_{i<j} w_{ijj}\binom{d(i)}{2}[x_{1j}, x_{1i}, x_{1i}, x_{1j}]\\ & + \sum_{i<j} w''_{iij}\binom{d(j)}{2}[x_{1j}, x_{1i}, x_{1j}, x_{1i}].\end{aligned}$$

Here by the Witt identity we have $[x_{1j}, x_{1i}, x_{1i}, x_{1j}] = [x_{1j}, x_{1i}, x_{1j}, x_{1i}]$ and by Lemma 6.4, the last two terms is equal to zero. Hence we have Lemma 6.9. Q.E.D.

Furthermore, we have

LEMMA 6.10.

$$2\sum_{i<j} u_{ij}[x_{1i}, x_{1j}^{d(j)}] + \sum_{i<j} u_{ij}\frac{d(j)}{d(i)}[x_{1i}^{d(i)}, x_{1j}, x_{1j}] - \sum_{i,k} v_{ik}[x_{2k}, x_{1i}^{d(i)}] = 0.$$

Proof. Eliminating $\sum_{i<j} u_{ij}[x_{1i}, p_3(x_{1j}^{d(j)})]$, $\sum_{i<j} u_{ij}(d(j)/d(i))[p_3(x_{1i}^{d(i)}), x_{1j}]$, and $\sum_{i,k} v'_{ik}[x_{2k}^{e(k)}, x_{1i}]$ in Lemmas 6.3 and 6.6, we have

$$\begin{aligned}
0 = &-2\sum_{i<j} u_{ij}[x_{1i}, x_{1j}^{d(j)}] - \sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}[x_{1i}, x_{1j}, x_{1i}] \\
&+ \sum_{i<j} u_{ij}\binom{d(j)}{2}[x_{1i}, x_{1j}, x_{1j}] + \sum_{i,k} v_{ik}\, d(i)[x_{2k}, x_{1i}] \\
&- \sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{3}[x_{1i}, x_{1j}, x_{1i}, x_{1i}] \\
&+ \sum_{i<j} u_{ij}\binom{d(j)}{3}[x_{1i}, x_{1j}, x_{1j}, x_{1j}] \\
&- \sum_{i<j} u_{ij}\frac{d(j)}{d(i)}[x_{1i}^{d(i)}, x_{1j}, x_{1j}] - \sum_{i\leqslant j\leqslant k} w_{ijk}[x_{1i}^{d(i)}, x_{1k}, x_{1j}] \\
&- \sum_{i\leqslant j\leqslant k} w_{ijk}[x_{1i}^{d(i)}, x_{1j}, x_{1k}] - \sum_{i<j\leqslant k} w'_{ijk}[x_{1j}^{d(j)}, x_{1k}, x_{1i}].
\end{aligned}$$

By expanding the index sets of the last three terms we obtain the equation

$$\begin{aligned}
0 = &\text{(the first seven terms)} \\
&- \sum_{i<j} w_{iij}[x_{1i}^{d(i)}, x_{1j}, x_{1i}] - 2\sum_{i<j} w_{ijj}[x_{1i}^{d(i)}, x_{1j}, x_{1j}] \\
&- \sum_{i<j<k} w_{ijk}[x_{1i}^{d(i)}, x_{1k}, x_{1j}] - \sum_{i<j<k} w_{ijk}[x_{1i}^{d(i)}, x_{1j}, x_{1k}] \\
&- \sum_{i<j<k} w'_{ijk}[x_{1j}^{d(j)}, x_{1k}, x_{1i}].
\end{aligned}$$

By Lemma 6.5(D_3), the second part of Lemma 6.8, and $w_{ijk} = -w'_{ijk}(d(j)/d(i)) - w''_{ijk}(d(k)/d(i))$ $(i<j<k)$, we have

$$\begin{aligned}
0 = &-2\sum_{i<j} u_{ij}[x_{1i}, x_{1j}^{d(j)}] - \sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}[x_{1i}, x_{1j}, x_{1i}] \\
&- \sum_{i<j} u_{ij}\binom{d(j)}{2}[x_{1i}, x_{1j}, x_{1j}] + \sum_{i,k} v_{ik}\, d(i)[x_{2k}, x_{1i}]
\end{aligned}$$

$$- \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]$$

$$- \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{3} [x_{1i}, x_{1j}, x_{1i}, x_{1i}] + \sum_{i<j} u_{ij} \binom{d(j)}{3} [x_{1i}, x_{1j}, x_{1j}, x_{1j}]$$

$$- \sum_{i<j<k} w'_{ijk} \left\{ \begin{aligned} &[x_{1j}^{d(j)}, x_{1k}, x_{1i}] + \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1k}, x_{1j}] - [x_{1j}^{d(j)}, x_{1k}, x_{1i}] \\ &- \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1k}] + 2 \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1k}] \\ &- 2[x_{1j}^{d(j)}, x_{1i}, x_{1k}] - \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1k}, x_{1j}] \end{aligned} \right\}$$

$$- \sum_{i<j<k} w''_{ijk} \left\{ \begin{aligned} &\frac{d(k)}{d(i)} [x_{1i}^{d(i)}, x_{1k}, x_{1j}] - [x_{1k}^{d(k)}, x_{1j}, x_{1i}] - \frac{d(k)}{d(i)} [x_{1i}^{d(i)}, x_{1k}, x_{1j}] \\ &+ 2 \frac{d(k)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1k}] - 2[x_{1k}^{d(k)}, x_{1i}, x_{1j}] \\ &- \frac{d(k)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1k}] \end{aligned} \right\}.$$

The above expressions in w'_{ijk} and w''_{ijk} simplify to

$$-3 \sum_{i<j<k} w'_{ijk} [x_{1i}, x_{1j}^{d(j)}, x_{1k}] + 3 \sum_{i<j<k} w''_{ijk} [x_{1k}^{d(k)}, x_{1i}, x_{1j}].$$

Substituting for these values from Lemma 6.9 and simplifying, we obtain

$$0 = -2 \sum_{i<j} u_{ij} [x_{1i}, x_{1j}^{d(j)}] + \sum_{i,k} v_{ik}\, d(i) [x_{2k}, x_{1i}] - \sum_{i,k} v_{ik} \binom{d(i)}{2} [x_{2k}, x_{1i}, x_{1i}]$$

$$- \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1j}] - \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{3} [x_{1i}, x_{1j}, x_{1i}, x_{1i}]$$

$$- \sum_{i<j} w_{iij} \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}, x_{1i}] + \sum_{i<j} u_{ij} \binom{d(j)}{3} [x_{1i}, x_{1j}, x_{1j}, x_{1j}]$$

$$+ \sum_{i<j} w'_{ijj} \binom{d(j)}{2} [x_{1i}, x_{1j}, x_{1j}, x_{1j}].$$

On the other hand, we have by Lemma 6.4

$$\sum_{i,k} v_{ik}\, d(i) [x_{2k}, x_{1i}] - \sum_{i,k} v_{ik} \binom{d(i)}{2} [x_{2k}, x_{1i}, x_{1i}] = \sum_{i,k} v_{ik} [x_{2k}, x_{1i}^{d(i)}],$$

$$\sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{3} [x_{1i}, x_{1j}, x_{1i}, x_{1i}] + \sum_{i<j} w_{iij} \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}, x_{1i}] = 0,$$

and

$$\sum_{i<j} u_{ij}\binom{d(j)}{3}[x_{1i}, x_{1j}, x_{1j}, x_{1j}] + \sum_{i<j} w'_{ijj}\binom{d(j)}{2}[x_{1i}, x_{1j}, x_{1j}, x_{1j}] = 0.$$

Therefore we have the desired equation. Q.E.D.

As its corollary of Theorem 6.1 we have

COROLLARY 6.11. *If G is a finite group with an N-series $G = H_1 \supseteq H_2 \supseteq \cdots \supset H_n \supseteq H_{n+1} \supseteq \cdots$, then the exponent of $G \cap \{1 + \Delta_5\}/H_5$ is divisible by $3!$. In particular the exponent of $D_5(G)/G_5$ is divisible by $3!$.*

Proof. We may assume that $H_5 = 1$. Let $g \in G \cap \{1 + \Delta_5\}$ be represented in the form in the proof of Theorem 6.1. Then we have

$$6g = 6\sum_{i<j}[x_{1i}, x_{1j}^{d(j)}] + 6\sum_{i<j<k} w'_{ijk}[x_{1i}, x_{1j}^{d(j)}, x_{1k}] - 6\sum_{i<j<k} w''_{ijk}[x_{1k}^{d(k)}, x_{1i}, x_{1j}]$$
$$+ 6\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}[x_{1i}, x_{1j}, x_{1i}] + 6\sum_{i,k} v_{ik}\left(\sum_{k<l} b_{il}[x_{2l}, x_{2k}]\right)$$

since $2\sum_{i<j} w_{ijj}\binom{d(i)}{2}[x_{1j}, x_{1i}, x_{1i}, x_{1j}] = 2\sum_{i<j} w_{iij}\binom{d(i)}{2}[x_{1i}, x_{1j}, x_{1i}, x_{1i}] = 0$. By Lemmas 6.9 and 6.10

$$6g = -3\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}[x_{1i}^{d(i)}, x_{1j}, x_{1j}] + 3\sum_{i,k} v_{ik}[x_{2k}, x_{1i}^{d(i)}]$$
$$-2\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}[x_{1i}, x_{1j}, x_{1i}] - 2\sum_{i<j} u_{ij}\binom{d(j)}{2}[x_{1i}, x_{1j}, x_{1j}]$$
$$+2\sum_{i,k} v_{ik}\binom{d(i)}{2}[x_{2k}, x_{1i}, x_{1i}] + 6\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}[x_{1i}, x_{1j}, x_{1i}]$$
$$+6\sum_{i,k} v_{ik}\left(\sum_{k<l} b_{il}[x_{2l}, x_{2k}]\right).$$

Now we obtain

$$2\sum_{i,k} v_{ik}\binom{d(i)}{2}[x_{2k}, x_{1i}, x_{1i}] = 2\sum_{i<j} w_{iij}\binom{d(i)}{2}[x_{1j}, x_{1i}, x_{1i}, x_{1j}]$$
$$= 2\sum_{i<j} w'_{ijj}\binom{d(i)}{2}[x_{1j}, x_{1i}, x_{1j}, x_{1j}] = 0,$$

and by Lemma 6.7

$$3\sum_{i,k} v_{ik}[x_{2k}, x_{1i}^{d(i)}] + 6\sum_{i,k} v_{ik}\left(\sum_{k<l} b_{il}[x_{2l}, x_{2k}]\right) = 0.$$

Hence

$$6g = -3 \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1j}] + 4 \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}]$$
$$- 2 \sum_{i<j} u_{ij} \binom{d(j)}{2} [x_{1i}, x_{1j}, x_{1j}].$$

Since $u_{ij}(d(j)/d(i))\binom{d(i)}{2} \equiv 0 \pmod{d(i)}$ by Theorem 6.1(A), we have

$$\sum_{i<j} u_{ij} \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]$$
$$= \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} \left\{ d(i)[x_{1i}, x_{1j}, x_{1j}] + \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}, x_{1j}] \right\}$$
$$= \sum_{i<j} u_{ij} d(j)[x_{1i}, x_{1j}, x_{1j}].$$

Furthermore, since $d(i)\, d(j)[x_{1i}, x_{1j}, x_{1i}] = 0 = d(j)^2[x_{1i}, x_{1j}, x_{1j}]$, we have

$$4u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{2} [x_{1i}, x_{1j}, x_{1i}] = 2u_{ij}\, d(j)(d(i) - 1)[x_{1i}, x_{1j}, x_{1i}]$$
$$= -2u_{ij}\, d(j)[x_{1i}, x_{1j}, x_{1i}]$$

and

$$2u_{ij} \binom{d(j)}{2} [x_{1i}, x_{1j}, x_{1j}] = u_{ij}\, d(j)(d(j) - 1)[x_{1i}, x_{1j}, x_{1j}]$$
$$= -u_{ij}\, d(j)[x_{1i}, x_{1j}, x_{1j}].$$

Hence

$$6g = -2 \sum_{i<j} u_{ij}\, d(j)[x_{1i}, x_{1j}, x_{1i}] - 2 \sum_{i<j} u_{ij}\, d(j)[x_{1i}, x_{1j}, x_{1j}].$$

By the second of Lemma 6.3

$$2 \sum_{i<j} u_{ij}\, d(j)[x_{1i}, x_{1j}, x_{1i}] + 2 \sum_{i<j} u_{ij}\, d(j)[x_{1i}, x_{1j}, x_{1j}] = 0$$

since $2 \sum_{i<j} u_{ij}\, d(j)[x_{1j}, x_{1i}, x_{1i}, x_{1j}] = 2 \sum_{i<j} u_{ij}\binom{d(j)}{2}[x_{1i}, x_{1j}, x_{1j}, x_{1i}] = 0$. Thus we have $6g = 0$.

COROLLARY 6.12. *Let G be any finite group with an N-series $G = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n \supseteq H_{n+1} \supseteq \cdots$. If G is $3!$-torsion free, then $G \cap \{1 + \Delta_k\} = H_k$ for $1 \leqslant k \leqslant 5$.*

It is well known that the exponent of $G \cap \{1 + \Delta_4\}/H_4$ is divisible by 2!, and we proved that the exponent of $G \cap \{1 + \Delta_5\}/H_5$ is divisible by 3!. So we present the following:

Problem. Let G be a finite group with an N-series $G = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n \supseteq H_{n+1} \supseteq \cdots$. Is the exponent of $G \cap \{1 + \Delta_n\}/H_n$ divisible by $(n-2)!$ for any $n \geqslant 2$?

If this problem is affirmatively proved for all $n \geqslant 2$, then it is a generalization of the well-known results of Cohn [1], Quillen [6], and Sjogren [8].

In a separate paper, we shall show that $D_5(G) = G_5 = 1$ for the counterexamples G to $D_4(G) = G_4$ which we constructed in [9], namely, $D_4(G) \supsetneqq G_4 = D_5(G) = G_5 = 1$ for these 2-groups G. Moreover, we shall construct counterexamples to $D_5(G) = G_5$.

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